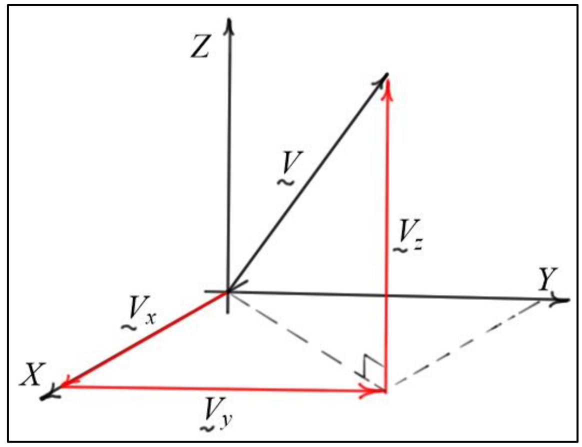
Elementary Statics
Vector Components and Vector Addition in Three Dimensions

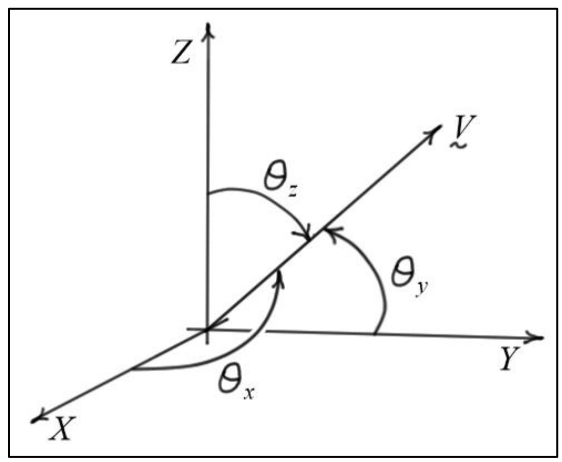
Cartesian Components of Vectors (3D)

- Given the ***magnitude*** of a vector and the ***direction*** of the vector relative to a set of three ***reference axes***, the vector can be expressed in terms of its ***components*** along those axes.
- For our convenience, it is beneficial to have the reference axes be a ***right-handed***, ***mutually perpendicular*** set.



- $\underset{\sim}{V}_x$, $\underset{\sim}{V}_y$, and $\underset{\sim}{V}_z$ represent the ***components*** of the vector $\underset{\sim}{V}$ along the ***right-handed***, ***mutually perpendicular*** *X*, *Y*, and *Z* axes. By vector addition, $\boxed{\underset{\sim}{V} = \underset{\sim}{V}_x + \underset{\sim}{V}_y + \underset{\sim}{V}_z}$
- The triangles formed by $\underset{\sim}{V}$ and $\underset{\sim}{V}_x$, by $\underset{\sim}{V}$ and $\underset{\sim}{V}_y$, and by $\underset{\sim}{V}$ and $\underset{\sim}{V}_z$ are all ***right triangles***.
- If the ***magnitude*** of the vector $\underset{\sim}{V}$ is $|\underset{\sim}{V}| = V$, and if the ***angles*** that $\underset{\sim}{V}$ makes with the *X*, *Y*, and *Z* axes are θ_x, θ_y, and θ_z, respectively, then

$$\boxed{\begin{aligned} \underset{\sim}{V} &= \underset{\sim}{V}_x + \underset{\sim}{V}_y + \underset{\sim}{V}_z = V_x \underset{\sim}{i} + V_y \underset{\sim}{j} + V_z \underset{\sim}{k} \\ &= \left(V\cos(\theta_x)\right)\underset{\sim}{i} + \left(V\cos(\theta_y)\right)\underset{\sim}{j} + \left(V\cos(\theta_z)\right)\underset{\sim}{k} \\ &= V\left(\cos(\theta_x)\underset{\sim}{i} + \cos(\theta_y)\underset{\sim}{j} + \cos(\theta_z)\underset{\sim}{k}\right) \end{aligned}}$$

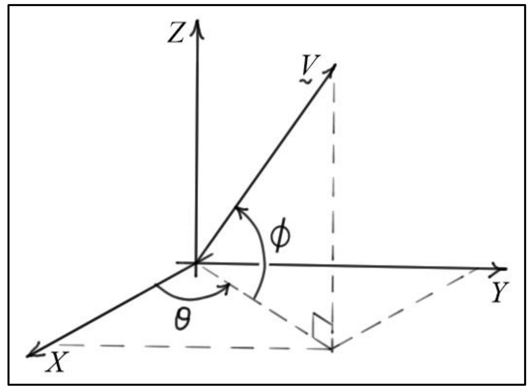


- The ***unit vectors*** $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$ indicate the positive *X*, *Y*, and *Z* coordinate directions.
- The vector $\boxed{\underset{\sim}{u}_V = \cos(\theta_x)\underset{\sim}{i} + \cos(\theta_y)\underset{\sim}{j} + \cos(\theta_z)\underset{\sim}{k}}$ is a ***unit vector*** in the direction of $\underset{\sim}{V}$.
- The angles θ_x, θ_y, and θ_z are ***not independent***. It can be shown that,

$$\boxed{\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1}$$

- The magnitude of $\underset{\sim}{V}$ is $\boxed{V = \sqrt{V_x^2 + V_y^2 + V_z^2}}$

Cartesian Components – Polar and Elevation Angles



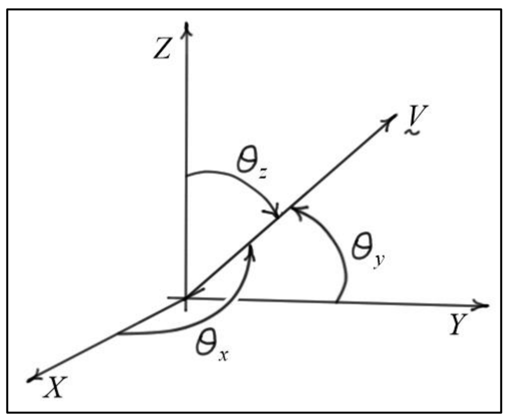
- The Cartesian components of a vector can also be given in terms of ***polar*** and ***elevation*** angles. Unlike the three angles relative to the *X*, *Y*, and *Z* axes, these two angles are ***independent***.
- In the diagram, angle θ measured in the *XY* plane is the ***polar angle***, and the angle ϕ is the ***elevation angle***.
- In this case, the Cartesian components are found using a two-step process. First, break $\underset{\sim}{V}$ into ***two components***, one in the *XY* plane and one perpendicular to it (along the *Z* axis). Then, break the component in the *XY* plane into two components, one along the *X* axis and one along the *Y* axis.

$$\boxed{\begin{aligned}\underset{\sim}{V} &= V\cos(\phi)\left(\cos(\theta)\,\underset{\sim}{i} + \sin(\theta)\,\underset{\sim}{j}\right) + V\sin(\phi)\,\underset{\sim}{k} \\ &= V\cos(\phi)\cos(\theta)\,\underset{\sim}{i} + V\cos(\phi)\sin(\theta)\,\underset{\sim}{j} + V\sin(\phi)\,\underset{\sim}{k} \\ &= V\left(\cos(\phi)\cos(\theta)\,\underset{\sim}{i} + \cos(\phi)\sin(\theta)\,\underset{\sim}{j} + \sin(\phi)\,\underset{\sim}{k}\right)\end{aligned}}$$

Example #1:

Given: Force $\underset{\sim}{F}$ has magnitude $|\underset{\sim}{F}| = 100$ (lb) and angles $\theta_x = 40$ (deg) and $\theta_y = 70$ (deg).

Find: Express the force $\underset{\sim}{F}$ in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$.



Solution:

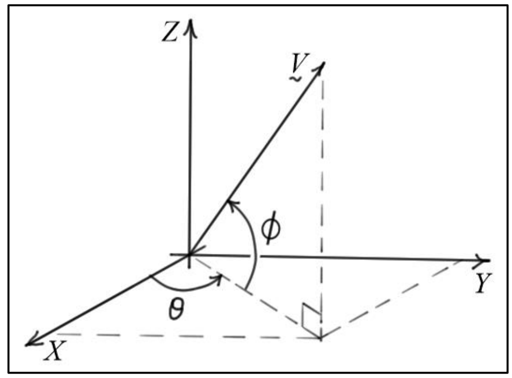
We know that: $\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1$. So,

$$\begin{aligned}\theta_z &= \cos^{-1}\left(\sqrt{1 - \cos^2(\theta_x) - \cos^2(\theta_y)}\right) \\ &= \cos^{-1}\left(\sqrt{1 - \cos^2(40) - \cos^2(70)}\right) \approx 57.027 \approx 57.0 \text{ (deg)}\end{aligned}$$

$$\boxed{\underset{\sim}{F} = 100\cos(40)\,\underset{\sim}{i} + 100\cos(70)\,\underset{\sim}{j} + 100\cos(57.027)\,\underset{\sim}{k} \approx 76.6\,\underset{\sim}{i} + 34.2\,\underset{\sim}{j} + 54.4\,\underset{\sim}{k} \text{ (lb)}}$$

Check: $\boxed{|\underset{\sim}{F}| \approx \sqrt{76.6^2 + 34.2^2 + 54.4^2} \approx 99.9828 \approx 100 \text{ (lb)}}$

Example #2: (polar and elevation angles)



Given: Force $\underset{\sim}{F}$ has magnitude $|\underset{\sim}{F}| = 100$ (lb) and angles $\theta = 24$ (deg) and $\phi = 33$ (deg).

Find: Express the force $\underset{\sim}{F}$ in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$.

Solution:

$$F_x = F\cos(\phi)\cos(\theta) = 100\cos(33)\cos(24) \approx 76.6164 \approx 76.6 \text{ (lbs)}$$

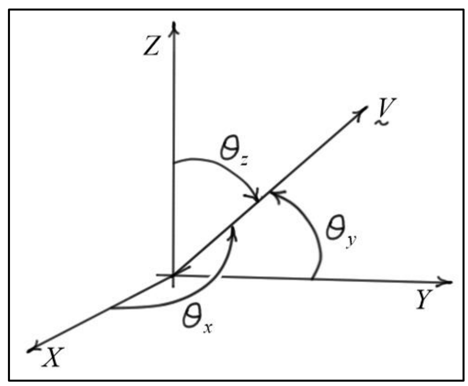
$$F_y = F\cos(\phi)\sin(\theta) = 100\cos(33)\sin(24) \approx 34.1118 \approx 34.1 \text{ (lbs)}$$

$$F_z = F\cos(\phi) = 100\sin(33) \approx 54.4639 \approx 54.5 \text{ (lbs)}$$

$$\Rightarrow \boxed{\underset{\sim}{F} \approx 76.6\underset{\sim}{i} + 34.1\underset{\sim}{j} + 54.5\underset{\sim}{k}}$$

Check: $\boxed{|\underset{\sim}{F}| \approx \sqrt{76.6^2 + 34.1^2 + 54.5^2} \approx 100.003 \approx 100 \text{ (lb)}}$

Example #3:



Given: A force $\underset{\sim}{F} = 70\,\underset{\sim}{i} + 50\,\underset{\sim}{j} - 80\underset{\sim}{k}$ (lb).

Find: The magnitude and direction of $\underset{\sim}{F}$ relative to the *X*, *Y*, and *Z* axes.

Solution:

$$\boxed{|\underset{\sim}{F}| = F = \sqrt{70^2 + 50^2 + 80^2} = 117.473 \approx 117 \text{ (lb)}}$$

$$\boxed{\theta_x = \cos^{-1}\left(F_x/F\right) \approx \cos^{-1}\left(70/117.473\right) \approx 53.425 \approx 53.4 \text{(deg)}}$$

$$\boxed{\theta_y = \cos^{-1}\left(F_y/F\right) \approx \cos^{-1}\left(50/117.473\right) \approx 64.81 \approx 64.8 \text{(deg)}}$$

$$\boxed{\theta_z = \cos^{-1}\left(F_z/F\right) \approx \cos^{-1}\left(-80/117.473\right) \approx 132.92 \approx 133 \text{(deg)}}$$

Check: $\boxed{\cos^2(53.43) + \cos^2(64.8) + \cos^2(132.9) \approx 0.9997 \approx 1}$

Vector Addition using Cartesian Components (3D)

To ***add two*** or ***more vectors***, simply express them in terms of the same unit vectors, and then add ***like components***. As before, we call the sum of the vectors the ***resultant***.

Example #4:

Given: $\underset{\sim}{F}_1 = 100\,\underset{\sim}{i} + 175\,\underset{\sim}{j} + 200\,\underset{\sim}{k}$ (lb)

$\underset{\sim}{F}_2 = -75\,\underset{\sim}{i} + 25\,\underset{\sim}{j} - 100\,\underset{\sim}{k}$ (lb)

$\underset{\sim}{F}_3 = -120\,\underset{\sim}{i} - 100\,\underset{\sim}{j} - 300\,\underset{\sim}{k}$ (lb)

Find: The magnitude and the direction of the resultant force $\underset{\sim}{F}$.

Solution:

$$\underset{\sim}{F} = (100 - 75 - 120)\,\underset{\sim}{i} + (175 + 25 - 100)\,\underset{\sim}{j} + (200 - 100 - 300)\,\underset{\sim}{k}$$

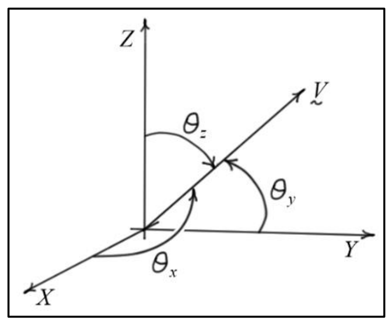
$$\Rightarrow \boxed{\underset{\sim}{F} = -95\,\underset{\sim}{i} + 100\,\underset{\sim}{j} - 200\,\underset{\sim}{k} \text{ (lb)}}$$

$$\boxed{|\underset{\sim}{F}| = F = \sqrt{95^2 + 100^2 + 200^2} \approx 242.951 \approx 243 \text{ (lb)}}$$

$$\boxed{\theta_x = \cos^{-1}(F_x/F) \approx \cos^{-1}(-95/242.951) \approx 113.018 \approx 113 \text{(deg)}}$$

$$\boxed{\theta_y = \cos^{-1}(F_y/F) \approx \cos^{-1}(100/242.951) \approx 65.6942 \approx 65.7 \text{(deg)}}$$

$$\boxed{\theta_z = \cos^{-1}(F_z/F) \approx \cos^{-1}(-200/242.951) \approx 145.408 \approx 145 \text{(deg)}}$$



Check: $\boxed{\cos^2(113.018) + \cos^2(65.6942) + \cos^2(145.408) \approx 1}$