Elementary Statics
Moment of a Force about an Axis

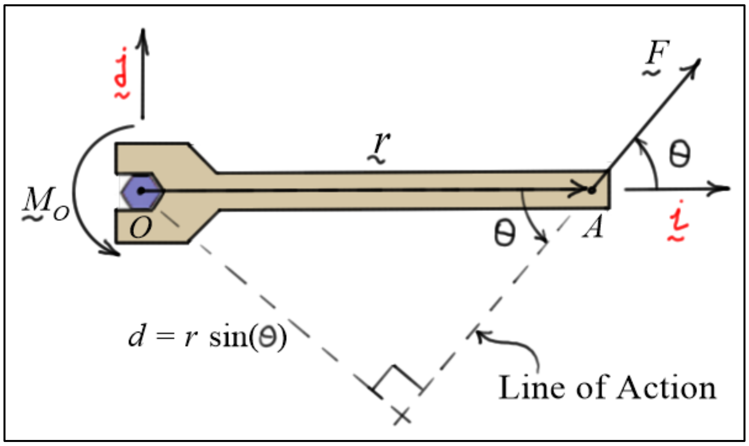
Two Dimensional Systems

- The ***moment*** of a force about a ***point*** O can be calculated using a ***cross product***.

$$\boxed{\underset{\sim}{M}_O = \underset{\sim}{r} \times \underset{\sim}{F}}$$

- Here, $\underset{\sim}{M}_O$ is ***perpendicular*** to the plane formed by the vectors $\underset{\sim}{r}$ and $\underset{\sim}{F}$, and it has ***magnitude*** $\boxed{|\underset{\sim}{M}_O| = |\underset{\sim}{F}|\, d = |\underset{\sim}{F}|\, r \sin(\theta)}$.

- In the ***two dimensional*** system shown, $\underset{\sim}{M}_O$ represents the ***moment*** of the force about an ***axis perpendicular*** to the plane of $\underset{\sim}{r}$ and $\underset{\sim}{F}$ (in the $\underset{\sim}{k}$ direction) and passing through point O.



Three Dimensional Systems

- To find the ***moment*** of a force about ***an axis*** in three-dimensional analysis, we first calculate the ***moment*** about ***any point on that axis***, say O, then we ***project*** that moment onto the axis using the dot product.

$$\boxed{M_{\underset{\sim}{n}\text{-axis}} = \underset{\sim}{M}_O \cdot \underset{\sim}{n} = \left(\underset{\sim}{r} \times \underset{\sim}{F}\right) \cdot \underset{\sim}{n}}$$

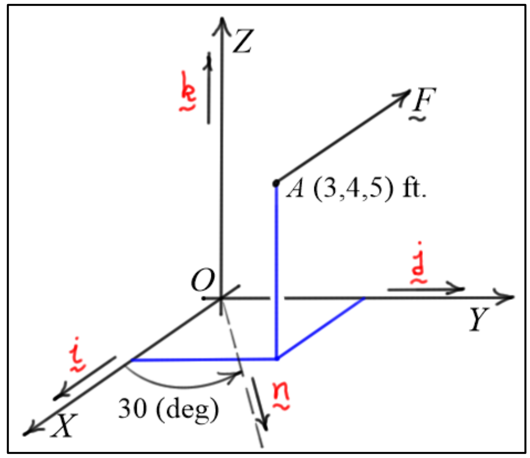
- As before, $\underset{\sim}{r}$ is a ***position vector*** from O to ***any point*** on the ***line of action*** of $\underset{\sim}{F}$.

- $M_{\underset{\sim}{n}\text{-axis}}$ is the ***scalar moment*** of $\underset{\sim}{F}$ about the ***axis*** passing through O and ***parallel*** to the ***unit vector*** $\underset{\sim}{n}$. In vector form, write $\boxed{\underset{\sim}{M}_{\underset{\sim}{n}\text{-axis}} = \left(\underset{\sim}{M}_O \cdot \underset{\sim}{n}\right)\underset{\sim}{n}}$.

- $M_{\underset{\sim}{n}\text{-axis}}$ can be ***positive*** or ***negative*** depending on the angle between $\underset{\sim}{n}$ and $\underset{\sim}{M}_O$. If it is ***positive***, point your right thumb in the direction of $\underset{\sim}{n}$ and your right fingers will show the circulation of $\underset{\sim}{F}$ about the axis. If it is ***negative***, point your right thumb opposite the direction of $\underset{\sim}{n}$ and your right fingers will show the circulation of $\underset{\sim}{F}$ about the axis.

- $M_{\underset{\sim}{n}\text{-axis}}$ is ***zero*** if $\underset{\sim}{F}$ is parallel to or intersects the axis. If $\underset{\sim}{F}$ is parallel to $\underset{\sim}{n}$, then $\underset{\sim}{r} \times \underset{\sim}{F}$ is perpendicular to $\underset{\sim}{n}$ making the dot product zero. If the line of action of $\underset{\sim}{F}$ intersects the axis, the position vector $\underset{\sim}{r}$ can be chosen to be zero.

<u>Example #1</u>:



Given: Force $\underset{\sim}{F} = -100\underset{\sim}{i} + 50\underset{\sim}{j} + 200\underset{\sim}{k}$ (lb) located at point A with coordinates $A:(3,4,5)$ (ft).

Find: a) M_x, M_y, and M_z the moments of $\underset{\sim}{F}$ about the X, Y, and Z axes, b) $M_{\underset{\sim}{n}\text{-axis}}$ the scalar moment of $\underset{\sim}{F}$ about an axis in the X-Y plane that makes an angle of 30 (deg) with the X-axis, and c) $\underset{\sim}{M}_{\underset{\sim}{n}\text{-axis}}$.

Solution:

a) $\underset{\sim}{M}_O = \underset{\sim}{r}_{A/O} \times \underset{\sim}{F}$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 4 & 5 \\ -100 & 50 & 200 \end{vmatrix} = (800-250)\underset{\sim}{i} - (600+500)\underset{\sim}{j} + (150+400)\underset{\sim}{k}$$

$$= 550\underset{\sim}{i} - 1100\underset{\sim}{j} + 550\underset{\sim}{k} \text{ (ft-lb)}$$

$$\boxed{M_x = \underset{\sim}{M}_O \cdot \underset{\sim}{i} = 550 \text{ (ft-lb)}}, \quad \boxed{M_y = \underset{\sim}{M}_O \cdot \underset{\sim}{j} = -1100 \text{ (ft-lb)}}, \quad \boxed{M_z = \underset{\sim}{M}_O \cdot \underset{\sim}{k} = 550 \text{ (ft-lb)}}$$

b) $\underset{\sim}{n} = \cos(30)\underset{\sim}{i} + \sin(30)\underset{\sim}{j}$

$$\boxed{M_{\underset{\sim}{n}\text{-axis}} = \underset{\sim}{M}_O \cdot \underset{\sim}{n} = \left(550\cos(30)\right) + \left(-1100 \cdot \sin(30)\right) + \left(550 \cdot 0\right) \approx -73.686 \approx -73.7 \text{ (ft-lb)}}$$

c) $\boxed{\underset{\sim}{M}_{\underset{\sim}{n}\text{-axis}} = \left(\underset{\sim}{M}_O \cdot \underset{\sim}{n}\right)\underset{\sim}{n} = -73.686\,\underset{\sim}{n} = -63.8\underset{\sim}{i} - 36.8\underset{\sim}{j} \text{ (ft-lb)}}$

<u>Note on Calculation of the Scalar Moment</u>: $M_{\underset{\sim}{n}\text{-axis}}$

- Calculation of the ***scalar moment*** can also be done in ***matrix determinant form***. Simply replace the first row of the determinant by the components of $\underset{\sim}{n}$ and expand as usual.

$$M_{\underset{\sim}{n}\text{-axis}} = \begin{vmatrix} n_x & n_y & n_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = n_x\left(r_y F_z - r_z F_y\right) - n_y\left(r_x F_z - r_z F_x\right) + n_z\left(r_x F_y - r_y F_x\right)$$

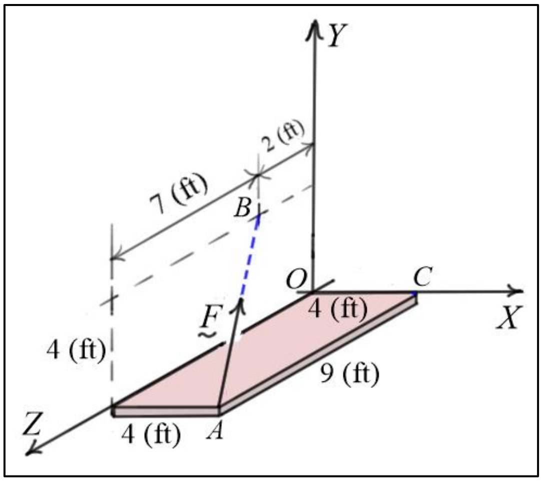
- So, for the Example #1, we have

$$M_{\underset{\sim}{n}\text{-axis}} = \begin{vmatrix} \cos(30) & \sin(30) & 0 \\ 3 & 4 & 5 \\ -100 & 50 & 200 \end{vmatrix} = \cos(30)(800-250) - \sin(30)(600+500)$$
$$\approx -73.7 \text{ (ft-lb)}$$

<u>Example #2</u>:

Given: Force $\underset{\sim}{F}$ is applied to the rectangular plate as shown. $\left|\underset{\sim}{F}\right| = 108$ (lb)

Find: M_x, M_y, and M_z the moments of $\underset{\sim}{F}$ about the X, Y, and Z axes as shown in the diagram.



Solution:

The unit vector pointing from A to B can be calculated using the dimensions shown in the figure as follows.

$$\underset{\sim}{u}_{AB} = \left(-4\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}\right) \Big/ \sqrt{4^2+4^2+7^2} = -\tfrac{4}{9}\underset{\sim}{i} + \tfrac{4}{9}\underset{\sim}{j} - \tfrac{7}{9}\underset{\sim}{k}$$

The force $\underset{\sim}{F}$ can then be written as

$$\underset{\sim}{F} = 108\underset{\sim}{u}_{AB} = 108\left(-\tfrac{4}{9}\underset{\sim}{i} + \tfrac{4}{9}\underset{\sim}{j} - \tfrac{7}{9}\underset{\sim}{k}\right) = -48\underset{\sim}{i} + 48\underset{\sim}{j} - 84\underset{\sim}{k}$$

The moments of $\underset{\sim}{F}$ about the X, Y, and Z axes can be calculated individually as follows.

$$M_x = \underset{\sim}{M}_O \cdot \underset{\sim}{i} = \left(\underset{\sim}{r}_{A/O} \times \underset{\sim}{F}\right)\cdot \underset{\sim}{i} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 9 \\ -48 & 48 & -84 \end{vmatrix} = -9(48) \quad \Rightarrow \boxed{M_x = -432 \text{ (ft-lb)}}$$

$$M_y = \underset{\sim}{M}_O \cdot \underset{\sim}{j} = \left(\underset{\sim}{r}_{A/O} \times \underset{\sim}{F}\right)\cdot \underset{\sim}{j} = \begin{vmatrix} 0 & 1 & 0 \\ 4 & 0 & 9 \\ -48 & 48 & -84 \end{vmatrix} = -\left[4(-84) - 9(-48)\right] \quad \Rightarrow \boxed{M_y = -96 \text{ (ft-lb)}}$$

$$M_z = \underset{\sim}{M}_O \cdot \underset{\sim}{k} = \left(\underset{\sim}{r}_{A/O} \times \underset{\sim}{F}\right) \cdot \underset{\sim}{k} = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 9 \\ -48 & 48 & -84 \end{vmatrix} = 4(48) \quad \Rightarrow \boxed{M_z = 192 \text{ (ft-lb)}}$$

Check: These moments can also be calculated as follows.

$$M_x = \underset{\sim}{M}_O \cdot \underset{\sim}{i} = \left(\underset{\sim}{r}_{B/O} \times \underset{\sim}{F}\right) \cdot \underset{\sim}{i} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ -48 & 48 & -84 \end{vmatrix} = 4(-84) - 2(48) \quad \Rightarrow \boxed{M_x = -432 \text{ (ft-lb)}}$$

$$M_y = \underset{\sim}{M}_O \cdot \underset{\sim}{j} = \left(\underset{\sim}{r}_{A/O} \times \underset{\sim}{F}\right) \cdot \underset{\sim}{j} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 4 & 2 \\ -48 & 48 & -84 \end{vmatrix} = -\left[0(-84) - 2(-48)\right] \quad \Rightarrow \boxed{M_y = -96 \text{ (ft-lb)}}$$

$$M_z = \underset{\sim}{M}_O \cdot \underset{\sim}{k} = \left(\underset{\sim}{r}_{A/O} \times \underset{\sim}{F}\right) \cdot \underset{\sim}{k} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 2 \\ -48 & 48 & -84 \end{vmatrix} = -4(-48) \quad \Rightarrow \boxed{M_z = 192 \text{ (ft-lb)}}$$