Elementary Statics
Static Equilibrium of a Rigid Body

- The diagram shows a ***rigid body*** under the action of a system of N ***forces***. Pairs of forces within the system may form ***couples***.
- For the body to be in ***static equilibrium*** (meaning that it remains ***stationary***), the following conditions must be met.

$$\boxed{\begin{aligned} \underset{\sim}{F}_R &= \sum_i \underset{\sim}{F}_i = \underset{\sim}{0} \\ \underset{\sim}{M}_P &= \sum_i \left(\underset{\sim}{p}_i \times \underset{\sim}{F}_i \right) = \underset{\sim}{0} \end{aligned}} \quad (P \text{ is any point})$$

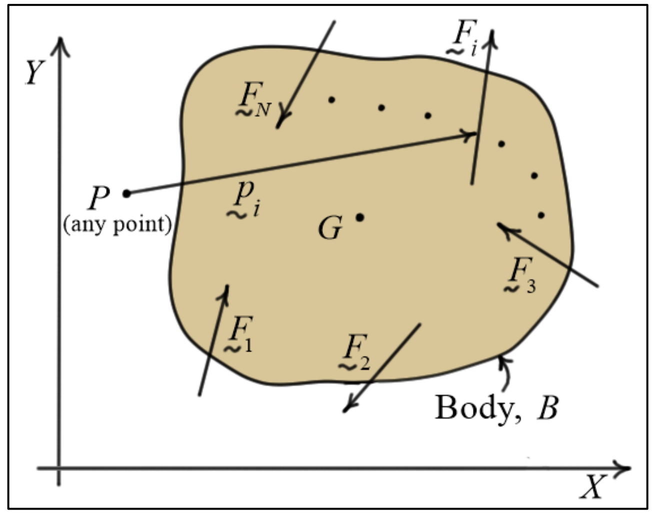


Fig. 1 Rigid Body with Applied Forces

- The ***first*** of these equations requires the ***sum*** of ***all forces acting*** on the body be ***zero***. Physically, this means the ***body will not translate***.
- The ***second*** set of these equations requires the ***sum of the moments of those forces*** about ***any point*** P also be ***zero***. Physically, this means the ***body will not rotate***.
- If a body cannot ***translate*** or ***rotate***, then it must ***remain stationary*** (in ***static equilibrium***).
- Note that the diagram shows the body ***free from its supports***, and as such, it is referred to as a ***free body diagram***.
- A ***free body diagram*** that ***depicts*** the ***correct nature*** of the ***forces acting on a body*** is critical in providing ***accurate*** and ***meaningful estimates*** of those forces.
- In ***two dimensional*** problems, the ***scalar equations of equilibrium*** can be written in ***any of the following three ways***. When using the ***second set*** of equations, the ***line passing through the points*** P and Q ***cannot be perpendicular to*** the chosen force summation direction. When using the ***third set***, the points P, Q, and R ***cannot be collinear***.

$$\boxed{\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_P &= 0 \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} &\sum F_x = 0 \quad \text{-or-} \quad \sum F_y = 0 \\ &\sum M_P = 0 \\ &\sum M_Q = 0 \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} \sum M_P &= 0 \\ \sum M_Q &= 0 \\ \sum M_R &= 0 \end{aligned}}$$

- ***Moment equations*** are often ***preferred***, because they allow us to ***solve more easily*** for the unknown forces or unknown moments.

Typical Supports

- ***Supports*** are used to keep a body in ***static equilibrium***, and to do so, they can ***apply forces*** and/or ***couples*** to the body. It is ***important*** when solving static equilibrium problems to be ***clear about the nature of these forces and couples***.
- Supports that ***restrict*** the ***translation*** of some point on the body ***apply a force*** to the body at that point and supports that ***restrict*** the ***rotation*** of the body ***apply a couple*** to the body at that point.

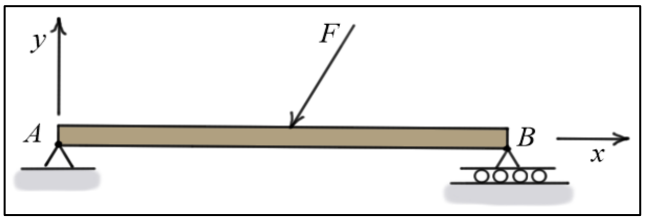


Fig. 2 Simply Supported Beam

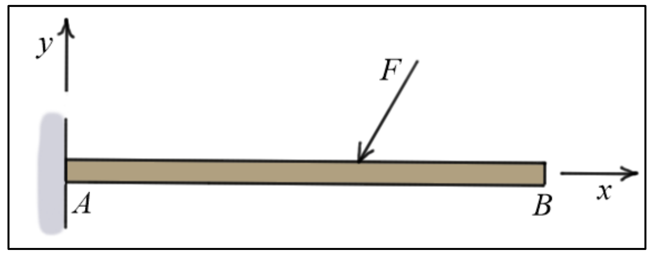


Fig. 3 Cantilevered Beam

- The support on the ***left end*** of the ***cantilevered beam*** is called a ***fixed support***. It restricts the ***movement*** of A in both the X and Y directions, and it restricts the ***rotation*** of the beam at A. To do so, it can produce ***forces*** in the X and Y directions and a ***couple moment*** in the Z direction.
- The support on the ***left end*** of the ***simply supported beam*** is called a ***pin support***. It restricts the ***movement*** of A in both the X and Y directions. To do so, it can produce ***forces in each of these directions***.
- The support on the ***right end*** of the ***simply supported beam*** is called a ***roller support***. It restricts the ***movement*** of B only in the ***negative*** Y direction. To do so, it can produce a ***force*** in the ***positive*** Y direction.
- The figure at the right depicts two members connected by a ***collar joint***. Assuming friction is negligible, the joint can produce ***forces*** in the Y and Z directions, but not in the X direction. It is also capable of producing ***couple moments*** in in the Y and Z directions, but not in the X direction.

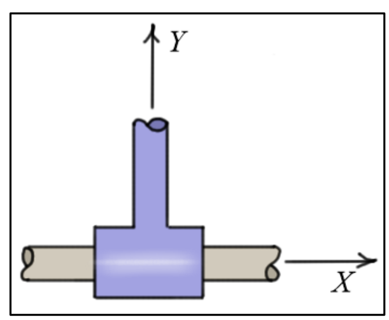


Fig. 4 Collar Joint

- There are ***many types of supports***. Refer to your ***textbook*** for a more ***detailed list***.

Sufficient Supports, Redundant Supports, and Improper Supports

- A body is considered to have ***sufficient supports*** if it has ***just enough*** supports to ***maintain its equilibrium***. That is, it has just enough supports to ***keep it from translating*** in any direction and to ***keep it from rotating about any axis***. In this case, the system is ***statically determinate***, meaning that we can find the support forces ***using the equations of statics alone***.
- A body has ***redundant supports*** if it has ***more than enough supports*** to maintain its equilibrium. In this case, the system is ***statically indeterminate***, meaning that we cannot find the support forces using the equations of statics alone. We need to include ***additional equations*** associated with the internal forces/displacements in the body.
- If a body is ***improperly supported***, then it ***does not have sufficient supports*** to maintain its equilibrium.

Two-Force and Three-Force Members

- If a body is acted upon by only ***two*** or ***three forces***, we can ***simplify*** the static equilibrium ***analysis***.
- If only ***two forces*** act on a body, then to satisfy equilibrium conditions, the forces must be ***equal in magnitude*** and ***opposite in direction***. Many ***structural members*** are taken to be two-force members if their weights can be ***neglected***.

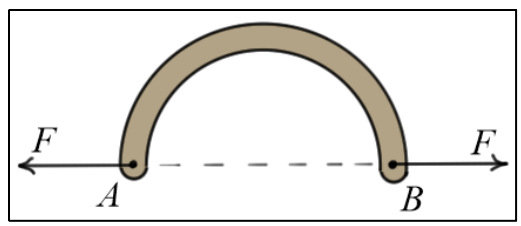


Fig. 5 Two Force Member

- If ***three forces*** act on a body, then the ***lines of actions*** of the forces must either all ***be parallel,*** or they must ***all intersect at a single point***.
- The diagram depicts a body of ***weight*** W being pushed along the floor by a ***force*** P. The ***force*** R represents the ***resultant*** of the ***distributed normal*** and ***friction forces*** exerted by the floor on the body. For the body to be in static equilibrium, the lines of action of the three forces must intersect at A.

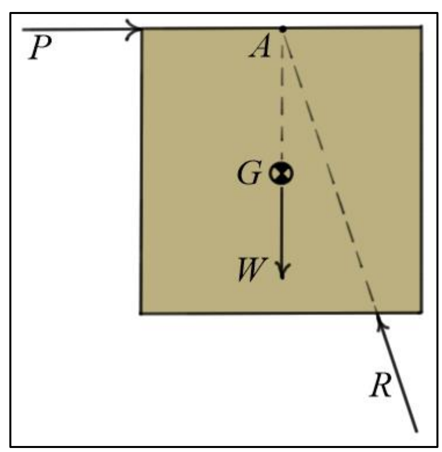
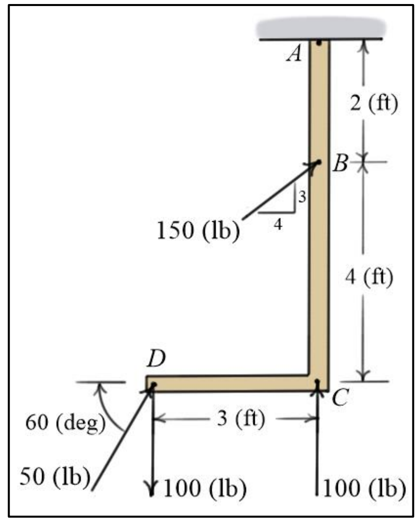


Fig. 6 Three-force Member

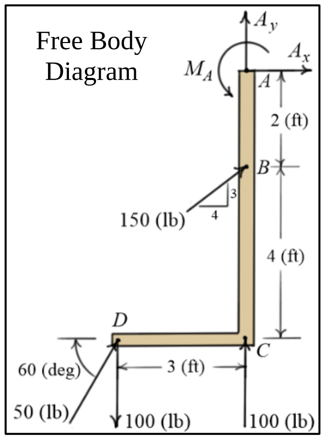
Example #1:

Given: L-shaped cantilevered bracket loaded as shown. Neglect the weight of the bracket.

Find: Force and moment ceiling applies to bracket at A.



Solution:



Equilibrium equations:

$$\sum F_x = A_x + \tfrac{4}{5}(150) + 50\cos(60) = 0$$

$$\Rightarrow \boxed{A_x = -145 \text{ (lb)}}$$

$$\sum F_y = A_y + \tfrac{3}{5}(150) + 50\sin(60) = 0$$

$$\Rightarrow \boxed{A_y \approx -133.3 \approx -133 \text{ (lb)}}$$

$$\sum M_A = M_A + \tfrac{4}{5}(150)(2) + (50\cos(60))(6) - (50\sin(60))(3) + (100)(3)$$

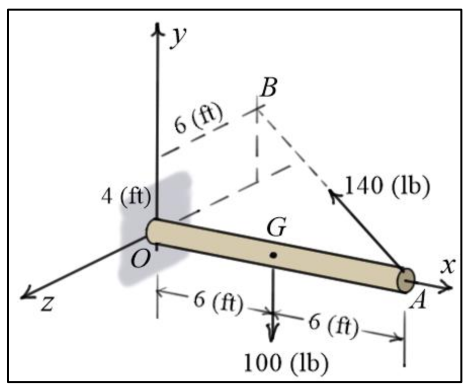
$$= M_A + 240 + 150 - 75\sqrt{3} + 300$$

$$\Rightarrow \boxed{M_A \approx -560.09 \approx -560 \text{ (ft-lb)}} \quad \Rightarrow \boxed{M_A \approx 560 \text{ (ft-lb) (clockwise)}}$$

Example #2:

Given: Cantilevered beam loaded as shown.

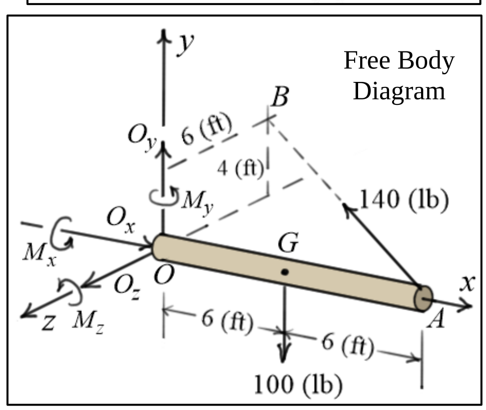
Find: Force and moment wall exerts on beam at O.



Solution:

$$\underset{\sim}{F}_A = 140\left(-12\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}\right) \Big/ \sqrt{12^2 + 4^2 + 6^2}$$

$$\Rightarrow \boxed{\underset{\sim}{F}_A = -120\underset{\sim}{i} + 40\underset{\sim}{j} - 60\underset{\sim}{k}}$$



Equilibrium Equations:

$$\sum F_x = O_x - 120 = 0 \quad \Rightarrow \boxed{O_x = 120 \text{ (lb)}}$$

$$\sum F_y = O_y + 40 - 100 = 0 \quad \Rightarrow \boxed{O_y = 60 \text{ (lb)}}$$

$$\sum F_z = O_z - 60 = 0 \quad \Rightarrow \boxed{O_z = 60 \text{ (lb)}}$$

$$\boxed{\underset{\sim}{F}_O = 120\underset{\sim}{i} + 60\underset{\sim}{j} + 60\underset{\sim}{k} \text{ (lb)}}$$

$$\sum \underset{\sim}{M}_O = \underset{\sim}{0} = \left[M_x \underset{\sim}{i} + M_y \underset{\sim}{j} + M_z \underset{\sim}{k} \right] + \left[\underset{\sim}{r}_{G/O} \times -100 \underset{\sim}{j} \right] + \left[\underset{\sim}{r}_{A/O} \times \underset{\sim}{F}_A \right]$$

$$= \left[M_x \underset{\sim}{i} + M_y \underset{\sim}{j} + M_z \underset{\sim}{k} \right] + \left[6 \underset{\sim}{i} \times -100 \underset{\sim}{j} \right] + \left[12 \underset{\sim}{i} \times \left(-120 \underset{\sim}{i} + 40 \underset{\sim}{j} - 60 \underset{\sim}{k} \right) \right]$$

$$= \left[M_x \underset{\sim}{i} + M_y \underset{\sim}{j} + M_z \underset{\sim}{k} \right] + \left[-600 \underset{\sim}{k} \right] + \left[480 \underset{\sim}{k} + 720 \underset{\sim}{j} \right]$$

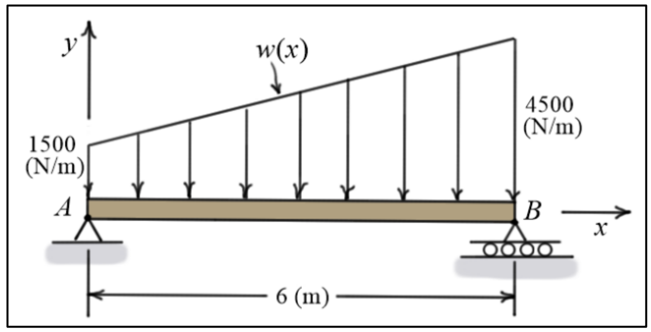
$$= M_x \underset{\sim}{i} + \left(M_y + 720 \right) \underset{\sim}{j} + \left(M_z - 600 + 480 \right) \underset{\sim}{k}$$

$$\Rightarrow \boxed{M_x = 0} \quad \boxed{M_y = -720 \text{ (ft-lb)}} \quad \boxed{M_z = 120 \text{ (ft-lb)}}$$

Example #3:

Given: Simply supported beam loaded as shown.

Find: Support forces at A and B.



Solution:

External Load:

Constant distributed load:

$$\boxed{\underset{\sim}{F}_1 = -6(1500) \underset{\sim}{j} = -9000 \underset{\sim}{j} = -9 \underset{\sim}{j} \text{ (kN)}}$$ acting at $\boxed{\bar{x}_1 = 3 \text{ (m)}}$ the midpoint of the beam

Triangular distributed load:

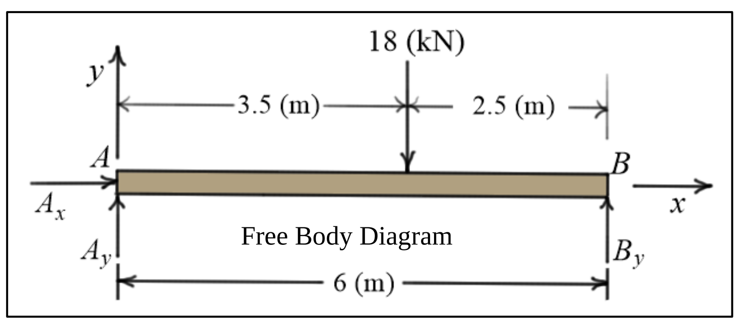
$$\boxed{\underset{\sim}{F}_2 = -\tfrac{1}{2}(6)(3000) j = -9000 \underset{\sim}{j} = -9 \underset{\sim}{j} \text{ (kN)}}$$ acting at $\boxed{\bar{x}_2 = \tfrac{2}{3}(6) = 4 \text{ (m)}}$.

Total load:

$$\boxed{\underset{\sim}{F}_R = \sum_{i=1}^{2} \underset{\sim}{F}_i = -18000 \underset{\sim}{j} \text{ (N)} = -18 \underset{\sim}{j} \text{ (kN)}}$$

$$\boxed{\bar{x} = \tfrac{1}{F_R}\left(F_1 \bar{x}_1 + F_2 \bar{x}_2 \right) = \tfrac{1}{18}\left(9(3) + 9(4) \right) = \tfrac{7}{2} = 3.5 \text{ (m)}}$$

Equilibrium Equations:



$$\boxed{\sum F_x = A_x = 0}$$

$$\sum M_A = 6B_y - 3.5(18) = 0$$

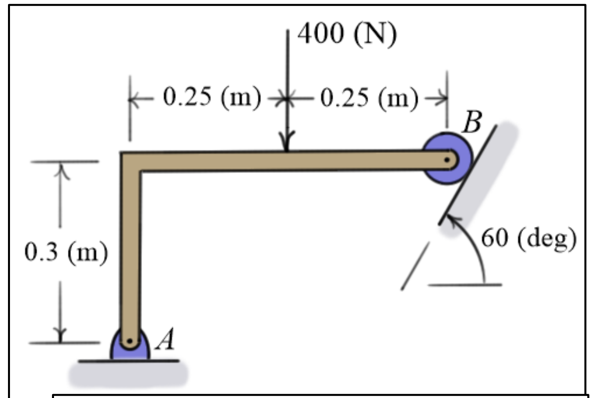
$$\Rightarrow \boxed{B_y = 10.5 \text{ (kN)}}$$

$$\sum F_y = A_y + B_y - 18 = 0 \quad \Rightarrow \boxed{A_y = 18 - B_y = 7.5 \text{ (kN)}}$$

Example #4:

Given: L-shaped bracket loaded as shown.
Neglect the weight of the bracket.

Find: Reaction forces at A and B



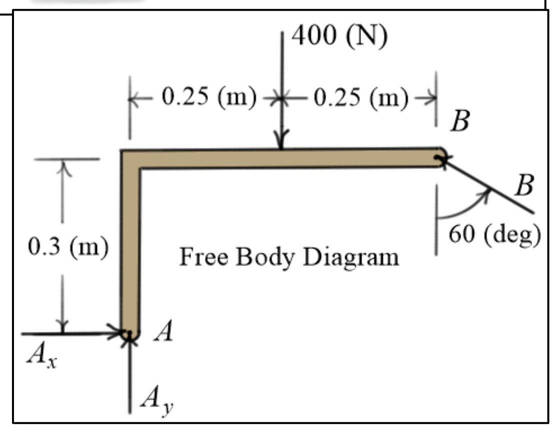
Solution:

Equilibrium Equations:

$$\boxed{\sum F_x = A_x - B\sin(60) = 0}$$

$$\boxed{\sum F_y = A_y + B\cos(60) - 400 = 0}$$

$$\boxed{\sum M_A = 0.5\cos(60)B + 0.3\sin(60)B - 0.25(400) = 0}$$



The moment equation can be simplified to give

$$B = \frac{0.25(400)}{0.5\cos(60) + 0.3\sin(60)} \approx \frac{100}{0.509808}$$

$$\Rightarrow \boxed{B \approx 196.152 \approx 196 \text{ (N)}}$$

Substituting back into the two force equations gives

$$\boxed{A_x = B\sin(60) \approx 169.87 \approx 170 \text{ (N)}} \quad \boxed{A_y = 400 - B\cos(60) \approx 301.924 \approx 302 \text{ (N)}}$$

Note that even though the load is vertical, the angle of the support at B induces horizontal reactions.