Multibody Dynamics
Coordinate Transformation (Rotation) Matrices

Relationships Between Unit Vectors in Different Reference Frames

The unit vectors of two mutually perpendicular unit vector sets $A:\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$ and $B:\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$ can be related using coordinate transformation matrices. To do this, write

$$\boxed{\underset{\sim}{e}_i = \sum_{j=1}^{3}\left(\underset{\sim}{e}_i \cdot \underset{\sim}{n}_j\right)\underset{\sim}{n}_j} \quad (i=1,2,3)$$

Or, in matrix form,

$$\{e\} = \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = \begin{bmatrix} \left(\underset{\sim}{e}_1 \cdot \underset{\sim}{n}_1\right) & \left(\underset{\sim}{e}_1 \cdot \underset{\sim}{n}_2\right) & \left(\underset{\sim}{e}_1 \cdot \underset{\sim}{n}_3\right) \\ \left(\underset{\sim}{e}_2 \cdot \underset{\sim}{n}_1\right) & \left(\underset{\sim}{e}_2 \cdot \underset{\sim}{n}_2\right) & \left(\underset{\sim}{e}_2 \cdot \underset{\sim}{n}_3\right) \\ \left(\underset{\sim}{e}_3 \cdot \underset{\sim}{n}_1\right) & \left(\underset{\sim}{e}_3 \cdot \underset{\sim}{n}_2\right) & \left(\underset{\sim}{e}_3 \cdot \underset{\sim}{n}_3\right) \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = [R]\{n\}$$

Here, $\boxed{R_{ij} = \underset{\sim}{e}_i \cdot \underset{\sim}{n}_j = \cos\left(\underset{\sim}{e}_i, \underset{\sim}{n}_j\right)}$ represents the ***cosine*** of the angle between the unit vectors $\underset{\sim}{e}_i$ and $\underset{\sim}{n}_j$. The 3×3 matrix $[R]$ is called the ***direction cosine matrix***. It can be shown that the matrix $[R]$ is ***orthogonal***, so its ***inverse*** is equal to its ***transpose***. Hence,

$$\boxed{\{e\} = [R]\{n\}} \quad \text{and} \quad \boxed{\{n\} = [R]^T\{e\}}$$

Relationships Between Vector Components in Different Reference Frames

Given the representations of a vector $\underset{\sim}{a}$ in two different reference frames

$$\boxed{\underset{\sim}{a} = \sum_{i=1}^{3} a_i \underset{\sim}{n}_i = \sum_{i=1}^{3} a'_i \underset{\sim}{e}_i},$$

the components a_i $(i=1,2,3)$ can be related to the components a'_i $(i=1,2,3)$ using transformation matrices as follows. Writing the above equation in matrix form

$$\boxed{\{a\}^T\{n\} = \{a'\}^T\{e\} = \{a'\}^T[R]\{n\}}$$

Comparing both sides of the equation, gives

$$\{a\}^T = \{a'\}^T[R] \quad \text{or} \quad \boxed{\{a\} = \left(\{a'\}^T[R]\right)^T = [R]^T\{a'\}}$$

Finally, using the fact that $[R]$ is ***orthogonal***

$$\boxed{\{a'\} = [R]\{a\}}$$

<u>Dot and Cross Products Revisited</u>

Transformation matrices can also be used to take the ***dot*** or ***cross*** products of two vectors expressed in two ***different*** reference frames as follows

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \quad \rightarrow \quad \boxed{\{a\}^T\{b\} = \{a\}^T[R]^T\{b'\}}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} \quad \rightarrow \quad \boxed{[\tilde{a}]\{b\} = [\tilde{a}][R]^T\{b'\}}$$

Recall that $[\tilde{a}]$ is the skew-symmetric matrix defined as

$$\boxed{[\tilde{a}] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}$$