

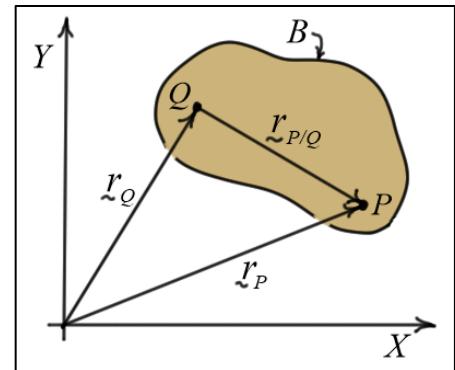
## Elementary Dynamics

### Relative Velocity of Two Points Fixed on a Rigid Body

The figure depicts a rigid body  $B$  moving in two dimensions. The two points  $P$  and  $Q$  are **fixed** on  $B$ . At any instant of time, the position vector of  $P$  can be written as

$$\underline{r}_P = \underline{r}_Q + \underline{r}_{P/Q}$$

Here,  $\underline{r}_{P/Q}$  is called the position vector of  $P$  **relative** to  $Q$ .



The **velocities** of  $P$  and  $Q$  can be related to each other by **differentiating** the above equation.

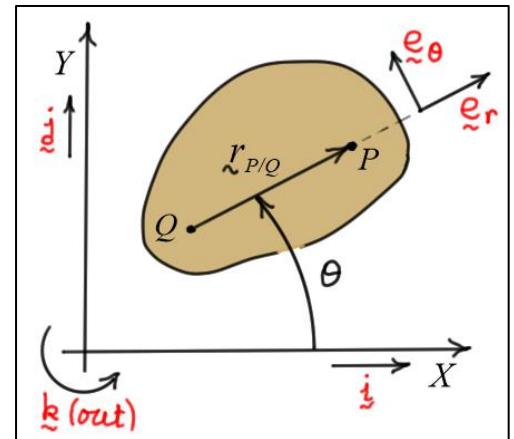
$$\frac{d}{dt}(\underline{r}_P) = \frac{d}{dt}(\underline{r}_Q) + \frac{d}{dt}(\underline{r}_{P/Q}) \quad \text{or} \quad \underline{v}_P = \underline{v}_Q + \frac{d}{dt}(\underline{r}_{P/Q})$$

Note because the body is rigid, the position vector  $\underline{r}_{P/Q}$  has **constant length**. However, the derivative of  $\underline{r}_{P/Q}$  is **not zero**, because it will **change direction** as the body rotates.

To calculate the derivative of  $\underline{r}_{P/Q}$ , consider the figure at the right. Given  $\underline{e}_r$  is a unit vector pointed from  $Q$  towards  $P$ , the position vector  $\underline{r}_{P/Q}$  can be written as

$$\underline{r}_{P/Q} = L \underline{e}_r$$

Here  $L$  represents the distance from  $Q$  to  $P$ . The derivative of  $\underline{r}_{P/Q}$  can be calculated as follows.



$$\frac{d}{dt}(\underline{r}_{P/Q}) = \frac{d}{dt}(L \underline{e}_r) = L \dot{\underline{e}}_r = L(\dot{\theta} \underline{k}) \times \underline{e}_r = \dot{\theta} \underline{k} \times L \underline{e}_r = \underline{\omega} \times \underline{r}_{P/Q} = L \dot{\theta} \underline{e}_\theta$$

In this equation,  $\underline{\omega} \triangleq \dot{\theta} \underline{k}$  is the **angular velocity** of the body. Combining this result with the boxed equation above gives

$$\underline{v}_P = \underline{v}_Q + (\underline{\omega} \times \underline{r}_{P/Q}) = \underline{v}_Q + \underline{v}_{P/Q} \quad \text{with} \quad \underline{v}_{P/Q} \triangleq \underline{\omega} \times \underline{r}_{P/Q}$$

This last equation defines  $\underline{v}_{P/Q}$  the **velocity** of  $P$  **relative** to  $Q$ . This equation is used to relate the velocity of **two points fixed** on the **same rigid body**.