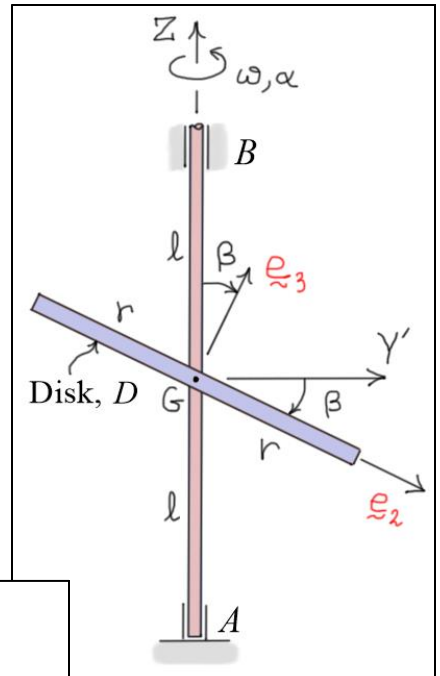
Intermediate Dynamics
Angular Momentum and Kinetic Energy of a Misaligned Disk



<u>Intermediate Dynamics Example 12:</u> (Disk welded to the shaft)

Reference frames: (R is a fixed frame)

S : $\underset{\sim}{i}'$, $\underset{\sim}{j}'$, $\underset{\sim}{k}$ (rotates with the shaft; aligned with the shaft)

D: $\underset{\sim}{i}'$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ (rotates with the shaft; aligned with the disk)

Find:

$\underset{\sim}{H}_G$ … ***angular momentum*** of the disk about its mass center, G

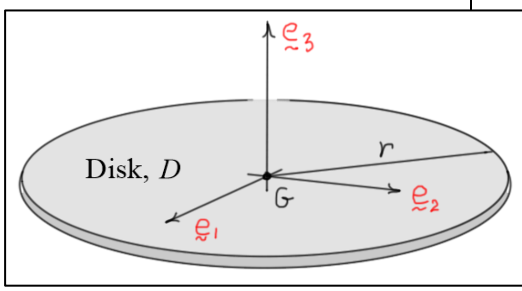
K … ***kinetic energy*** of the disk

Solution:

Expressing $\underset{\sim}{H}_G$ in the disk reference frame D:

$$\boxed{{}^R\underset{\sim}{\omega}_D = \omega\,\underset{\sim}{k} = \omega\left(-S_\beta\,\underset{\sim}{e}_2 + C_\beta\,\underset{\sim}{e}_3\right)}$$

$$\boxed{\left[I_G\right]_D = mr^2\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}$$



$$\begin{Bmatrix} \underset{\sim}{H}_G\cdot\underset{\sim}{i}' \\ \underset{\sim}{H}_G\cdot\underset{\sim}{e}_2 \\ \underset{\sim}{H}_G\cdot\underset{\sim}{e}_3 \end{Bmatrix} = mr^2\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} 0 \\ -\omega S_\beta \\ \omega C_\beta \end{Bmatrix} = mr^2\begin{Bmatrix} 0 \\ -\frac{1}{4}\omega S_\beta \\ \frac{1}{2}\omega C_\beta \end{Bmatrix} \quad\Rightarrow\quad \boxed{\underset{\sim}{H}_G = \tfrac{1}{4}mr^2\omega\left(-S_\beta\,\underset{\sim}{e}_2 + 2C_\beta\,\underset{\sim}{e}_3\right)}$$

Expressing $\underset{\sim}{H}_G$ in the shaft reference frame S:

$$\underset{\sim}{H}_G = \tfrac{1}{4}mr^2\omega\left(-S_\beta\,\underset{\sim}{e}_2 + 2C_\beta\,\underset{\sim}{e}_3\right) = \tfrac{1}{4}mr^2\omega\left[-S_\beta\left(C_\beta\,\underset{\sim}{j}' - S_\beta\,\underset{\sim}{k}\right) + 2C_\beta\left(S_\beta\,\underset{\sim}{j}' + C_\beta\,\underset{\sim}{k}\right)\right]$$

Or,

$$\boxed{\underset{\sim}{H}_G = \tfrac{1}{4}mr^2\omega\left[\left(S_\beta C_\beta\right)\underset{\sim}{j}' + \left(2C_\beta^2 + S_\beta^2\right)\underset{\sim}{k}\right] = \tfrac{1}{4}mr^2\omega\left[\left(S_\beta C_\beta\right)\underset{\sim}{j}' + \left(C_\beta^2 + 1\right)\underset{\sim}{k}\right]}$$

Also,

$$\boxed{\underset{\sim}{H}_G = \left(-I_{X'Z}\,\omega\right)\underset{\sim}{i}' + \left(-I_{Y'Z}\,\omega\right)\underset{\sim}{j}' + \left(I_{ZZ}\,\omega\right)\underset{\sim}{k}}$$

Comparing these two results gives: $\boxed{I_{X'Z} = 0}$, $\boxed{I_{Y'Z} = -\tfrac{1}{4}mr^2 S_\beta C_\beta}$, and $\boxed{I_{ZZ} = \tfrac{1}{4}mr^2\left(C_\beta^2 + 1\right)}$

The ***kinetic energy*** of the disk is found from the ***velocity*** and ***angular momentum*** to be

$$K = \underbrace{\tfrac{1}{2}m({}^R\underset{\sim}{v}_G)^2}_{\text{zero}} + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_D\cdot\underset{\sim}{H}_G = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_D\cdot\underset{\sim}{H}_G = \tfrac{1}{2}(\omega\underset{\sim}{k})\cdot\underset{\sim}{H}_G = \boxed{\tfrac{1}{8}mr^2\left(C_\beta^2 + 1\right)\omega^2}$$