**Elementary Dynamics**
Mass Moments of Inertia in Two Dimensions

Definition of Mass Moment of Inertia

The figure depicts a rigid body in two dimensions. The moment of inertia of the body about the Z-axis passing through the point O is defined as

$$\boxed{\left(I_Z\right)_O = \int_B r^2 dm = \int_B \left(x^2 + y^2\right) dm}$$

The moment of inertia $\left(I_Z\right)_O$ measures the distribution of mass about the Z-axis.
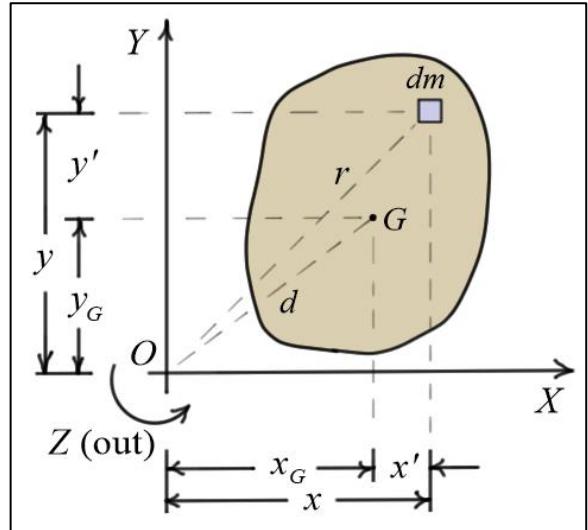
The *larger* the inertia, the *further* the mass is spread away from the axis. The *smaller* the inertia, the *closer* the mass is located to the axis. The units of I_Z are ***slug-ft²*** or ***kg-m²***.

Parallel Axes Theorem

The moment of inertia of the body about the Z-axis can be related to the moment of inertia about an axis parallel to Z and passing through the mass center G as follows:

$$\left(I_Z\right)_O = \int_B \left(x^2 + y^2\right) dm = \int_B \left(\left(x_G + x'\right)^2 + \left(y_G + y'\right)^2\right) dm$$

$$= \int_B \left(x_G^2 + y_G^2\right) dm + \int_B 2 x_G x' dm + \int_B 2 y_G y' dm + \int_B \left(x'^2 + y'^2\right) dm$$

$$= \left(x_G^2 + y_G^2\right) \underbrace{\int_B dm}_{M} + 2 x_G \underbrace{\left(\int_B x' dm\right)}_{\substack{\text{zero by} \\ \text{definition} \\ \text{of mass center}}} + 2 y_G \underbrace{\left(\int_B y' dm\right)}_{\substack{\text{zero by} \\ \text{definition} \\ \text{of mass center}}} + \left(I_Z\right)_G$$

$$= M d^2 + \left(I_Z\right)_G$$

or

$$\boxed{\left(I_Z\right)_O = \left(I_Z\right)_G + M d^2} \qquad \text{or} \qquad \boxed{I_O = I_G + M d^2} \qquad \text{(Parallel Axes Theorem)}$$

Mass Moments of Inertia for Composite Shapes

The mass moment of inertia of a composite shape is the sum of the inertias of the individual component shapes. If G is the mass center of the composite shape, then

$$I_G = \sum_i (I_G)_i$$

Here, $(I_G)_i$ represents the moment of inertia of the ith component shape about the composite mass center G and can be calculated as follows

$$(I_G)_i = (I_{G_i}) + m_i d_i^2.$$
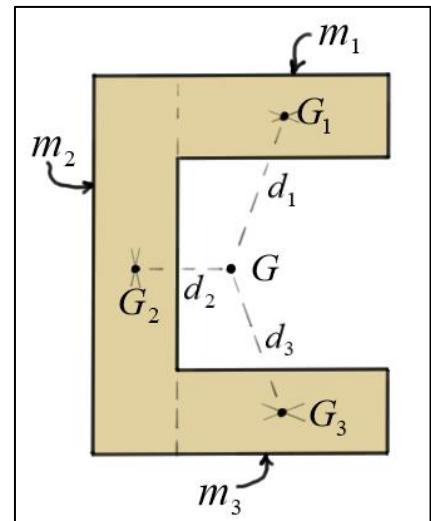
Here, I_{G_i} represents the moment of inertia of the ith component shape about its own mass center G_i and can be found in the tables for common geometric shapes.