

Elementary Dynamics

Angular Momentum of a Rigid Body in Two Dimensions

The figure depicts a rigid body B moving in two dimensions. The XY axes are fixed, the xy axes are fixed in and rotate with B . G is the mass center of B . The **angular momentum** of the body about G is defined as

$$H_G = \int_B (\underline{r} \times \underline{v}_P) dm$$

Using the **relative velocity equation**, the angular momentum can be rewritten as follows.

$$H_G = \int_B \underline{r} \times (\underline{v}_G + \underline{v}_{P/G}) dm = \int_B (\underline{r} \times \underline{v}_G) dm + \int_B (\underline{r} \times \underline{v}_{P/G}) dm = \underbrace{\left(\int_B \underline{r} dm \right)}_{\text{zero by definition of mass center}} \times \underline{v}_G + \int_B \underline{r} \times (\omega \times \underline{r}) dm$$

$$\Rightarrow H_G = \int_B \underline{r} \times (\omega \times \underline{r}) dm$$

The triple vector product integrand can be written in component form as follows.

$$\omega \times \underline{r} = \omega \underline{k} \times (x' \underline{e}_1 + y' \underline{e}_2) = \omega (-y' \underline{e}_1 + x' \underline{e}_2)$$

$$\underline{r} \times (\omega \times \underline{r}) = \omega \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{k} \\ x' & y' & 0 \\ -y' & x' & 0 \end{vmatrix} = \omega (x'^2 + y'^2) \underline{k}$$

Substituting into the boxed equation above gives

$$H_G = \int_B \omega (x'^2 + y'^2) \underline{k} dm = \left(\int_B (x'^2 + y'^2) dm \right) \omega \underline{k} = I_G \omega \underline{k}$$

So, the **angular momentum** of a rigid body about its **mass center** can be written as follows.

$$H_G = I_G \omega$$

I_G is the mass moment of inertia of the body about its mass center G .)

Note: Recall the **linear momentum** of a rigid body is $\underline{L} = m \underline{v}_G$.

