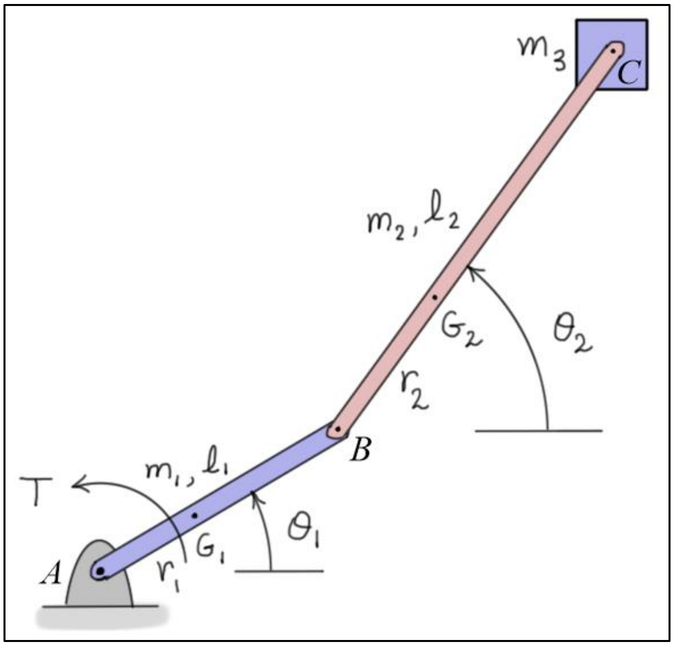
Intermediate Dynamics
Equations of Motion of a Slider-Crank Mechanism



The equations of motion of a ***slider-crank mechanism*** can be formulated in various ways. Here, the equations of motion are formulated for the system shown in the diagram. The necessary constraints to form the slider-crank mechanism are imposed using Lagrange's equations with Lagrange multipliers. The system shown consists of ***two links*** and an ***end mass***. A torque $\underset{\sim}{T}$ drives link AB on the shaft at A. It is assumed that the ***end mass*** m_3 ***translates but does not rotate***.

Constraint

The system shown can be converted into a simple ***slider-crank mechanism*** with ***zero offset***, by imposing the following configuration constraint.

$$\boxed{\ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = 0} \qquad (1)$$

This constraint can be put into ***standard form*** by ***differentiating*** it with respect to ***time*** to get

$$\boxed{\left(\ell_1 \cos(\theta_1)\right)\dot{\theta}_1 + \left(\ell_2 \cos(\theta_2)\right)\dot{\theta}_2 = 0} \qquad (2)$$

Equations of Motion

The equations of motion of the slider-crank mechanism can be developed using ***Lagrange's equations*** with a single ***Lagrange multiplier***. That is,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = F_{\theta_i} + \lambda_1 a_{1i}} \qquad (i = 1, 2) \qquad (3)$$

The coefficients a_{1i} $(i = 1, 2)$ are found by comparing Eq. (2) with the standard constraint equation form.

$$\boxed{a_{11} = \ell_1 \cos(\theta_1)} \qquad \boxed{a_{12} = \ell_2 \cos(\theta_2)} \qquad (4)$$

Including the ***kinetic energies*** of the ***three bodies*** and the ***potential energies*** associated with the ***weight forces***, it can be shown that

$$\boxed{\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right)-\frac{\partial L}{\partial\theta_1}=&\left(m_1r_1^2+I_1+m_2\ell_1^2+m_3\ell_1^2\right)\ddot\theta_1+\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\cos(\theta_2-\theta_1)\ddot\theta_2\\&-\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\sin(\theta_2-\theta_1)\,\dot\theta_2^2+\left(m_1r_1+m_2\ell_1+m_3\ell_1\right)g\cos(\theta_1)\end{aligned}} \quad (5)$$

$$\boxed{\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right)-\frac{\partial L}{\partial\theta_2}=&\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\cos(\theta_2-\theta_1)\,\ddot\theta_1+\left(m_2r_2^2+I_2+m_3\ell_2^2\right)\ddot\theta_2\\&+\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\sin(\theta_2-\theta_1)\,\dot\theta_1^2+\left(m_2r_2+m_3\ell_2\right)g\cos(\theta_2)\end{aligned}} \quad (6)$$

The ***contribution*** of the ***driving torque*** to the equations of motion are

$$\boxed{F_{\theta_1}=T\underset{\sim}{k}\cdot\left[\partial\underset{\sim}{\omega}_1\big/\partial\dot\theta_1\right]=T} \qquad \boxed{F_{\theta_2}=T\underset{\sim}{k}\cdot\left[\partial\underset{\sim}{\omega}_1\big/\partial\dot\theta_2\right]=0} \quad (7)$$

Using these results in the equations of motion (Eq. (3)) gives the equations of motion.

$$\boxed{\begin{aligned}&\left(m_1r_1^2+I_1+m_2\ell_1^2+m_3\ell_1^2\right)\ddot\theta_1+\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\cos(\theta_2-\theta_1)\ddot\theta_2\\&-\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\sin(\theta_2-\theta_1)\,\dot\theta_2^2+\left(m_1r_1+m_2\ell_1+m_3\ell_1\right)g\cos(\theta_1)=T+\lambda_1\ell_1\cos(\theta_1)\end{aligned}} \quad (8)$$

$$\boxed{\begin{aligned}&\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\cos(\theta_2-\theta_1)\,\ddot\theta_1+\left(m_2r_2^2+I_2+m_3\ell_2^2\right)\ddot\theta_2\\&+\left(m_2\ell_1r_2+m_3\ell_1\ell_2\right)\sin(\theta_2-\theta_1)\,\dot\theta_1^2+\left(m_2r_2+m_3\ell_2\right)g\cos(\theta_2)=\lambda_1\ell_2\cos(\theta_2)\end{aligned}} \quad (9)$$

Eqs. (8) and (9) must be solved along with the constraint equation to find θ_1, θ_2, and λ_1. Differentiating Eq. (2) gives

$$\boxed{\left(\ell_1\cos(\theta_1)\right)\ddot\theta_1+\left(\ell_2\cos(\theta_2)\right)\ddot\theta_2-\left(\ell_1\sin(\theta_1)\right)\dot\theta_1^2-\left(\ell_2\sin(\theta_2)\right)\dot\theta_2^2=0} \quad (10)$$

Eqs. (8), (9), and (10) form a set of ***three coupled, second-order, differential/algebraic equations*** that can be solved for the ***three unknowns*** θ_1, θ_2, and λ_1 as functions of time given the ***driving torque*** T and an ***initial position***.

The approach taken here represents just ***one way*** of breaking the system down and then putting it back together with configuration constraints. Question: What other systems could be joined to form the slider-crank mechanism?