Multibody Dynamics
Exercises #6 Answers

1. a) $\left(\frac{1}{3}mL^2\right)\ddot{\theta}+\frac{1}{2}mgL\sin(\theta)=0$

   b) Three differential/algebraic equations from Lagrange's equations

$$m\ddot{x}=\lambda_1$$
$$m\ddot{y}+mg=\lambda_2$$
$$\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}=\left(-\tfrac{1}{2}L\cos(\theta)\right)\lambda_1+\left(-\tfrac{1}{2}L\sin(\theta)\right)\lambda_2$$

   Two differentiated constraint equations

$$\ddot{x}-\left(\tfrac{1}{2}L\cos(\theta)\right)\ddot{\theta}+\left(\tfrac{1}{2}L\sin(\theta)\right)\dot{\theta}^2=0$$
$$\ddot{y}-\left(\tfrac{1}{2}L\sin(\theta)\right)\ddot{\theta}-\left(\tfrac{1}{2}L\cos(\theta)\right)\dot{\theta}^2=0$$

2. a) $\left(\frac{4}{3}mL^2\right)\ddot{\theta}_1+\left(\frac{1}{2}mL^2\cos(\theta_2-\theta_1)\right)\ddot{\theta}_2-\left(\frac{1}{2}mL^2\sin(\theta_2-\theta_1)\right)\dot{\theta}_2^2+\left(\frac{3}{2}mgL\right)\sin(\theta_1)=0$

   $\left(\frac{1}{2}mL^2\cos(\theta_2-\theta_1)\right)\ddot{\theta}_1+\left(\frac{1}{3}mL^2\right)\ddot{\theta}_2+\left(\frac{1}{2}mL^2\sin(\theta_2-\theta_1)\right)\dot{\theta}_1^2+\left(\frac{1}{2}mgL\right)\sin(\theta_2)=0$

   b) Six differential/algebraic equations from Lagrange's equations

$$m\ddot{x}_1=\lambda_1$$
$$m\ddot{y}_1+mg=\lambda_2$$
$$\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}_1=\left(-\tfrac{1}{2}L\cos(\theta_1)\right)\lambda_1+\left(-\tfrac{1}{2}L\sin(\theta_1)\right)\lambda_2+\left(-L\cos(\theta_1)\right)\lambda_3+\left(-L\sin(\theta_1)\right)\lambda_4$$
$$m\ddot{x}_2=\lambda_3$$
$$m\ddot{y}_2+mg=\lambda_4$$
$$\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}_2=\left(-\tfrac{1}{2}L\cos(\theta_2)\right)\lambda_3+\left(-\tfrac{1}{2}L\sin(\theta_2)\right)\lambda_4$$

   Four differentiated constraint equations

$$\ddot{x}_1+\left(-\tfrac{1}{2}L\cos(\theta_1)\right)\ddot{\theta}_1+\left(\tfrac{1}{2}L\sin(\theta_1)\right)\dot{\theta}_1^2=0$$
$$\ddot{y}_1+\left(-\tfrac{1}{2}L\sin(\theta_1)\right)\ddot{\theta}_1-\left(\tfrac{1}{2}L\cos(\theta_1)\right)\dot{\theta}_1^2=0$$
$$\ddot{x}_2+\left(-L\cos(\theta_1)\right)\ddot{\theta}_1+\left(L\sin(\theta_1)\right)\dot{\theta}_1^2+\left(-\tfrac{1}{2}L\cos(\theta_2)\right)\ddot{\theta}_2+\left(\tfrac{1}{2}L\sin(\theta_2)\right)\dot{\theta}_2^2=0$$
$$\ddot{y}_2+\left(-L\sin(\theta_1)\right)\ddot{\theta}_1-\left(L\cos(\theta_1)\right)\dot{\theta}_1^2+\left(-\tfrac{1}{2}L\sin(\theta_2)\right)\ddot{\theta}_2-\left(\tfrac{1}{2}L\cos(\theta_2)\right)\dot{\theta}_2^2=0$$

3. a) Two differential/algebraic equations from Lagrange's equations

$$\left(\tfrac{4}{3}mL^2\right)\ddot{\theta}_1+\left(\tfrac{1}{2}mL^2\cos(\theta_2-\theta_1)\right)\ddot{\theta}_2-\left(\tfrac{1}{2}mL^2\sin(\theta_2-\theta_1)\right)\dot{\theta}_2^2+\left(\tfrac{3}{2}mgL\right)\sin(\theta_1)=M+\lambda_1$$
$$\left(\tfrac{1}{2}mL^2\cos(\theta_2-\theta_1)\right)\ddot{\theta}_1+\left(\tfrac{1}{3}mL^2\right)\ddot{\theta}_2+\left(\tfrac{1}{2}mL^2\sin(\theta_2-\theta_1)\right)\dot{\theta}_1^2+\left(\tfrac{1}{2}mgL\right)\sin(\theta_2)=\lambda_1$$

   One differentiated constraint equation

$$\ddot{\theta}_1+\ddot{\theta}_2=0$$

b) Six differential/algebraic equations from Lagrange's equations

$$m\ddot{x}_1 = \lambda_1$$

$$m\ddot{y}_1 + mg = \lambda_2$$

$$\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}_1 = M + \left(-\tfrac{L}{2}\cos(\theta_1)\right)\lambda_1 + \left(-\tfrac{L}{2}\sin(\theta_1)\right)\lambda_2 + \left(-L\cos(\theta_1)\right)\lambda_3 + \left(-L\sin(\theta_1)\right)\lambda_4 + \lambda_5$$

$$m\ddot{x}_2 = \lambda_3$$

$$m\ddot{y}_2 + mg = \lambda_4$$

$$\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}_2 = \left(-\tfrac{L}{2}\cos(\theta_2)\right)\lambda_3 + \left(-\tfrac{L}{2}\sin(\theta_2)\right)\lambda_4 + \lambda_5$$

Five differentiated constraint equations

$$\ddot{x}_1 + \left(-\tfrac{1}{2}L\cos(\theta_1)\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}L\sin(\theta_1)\right)\dot{\theta}_1^2 = 0$$

$$\ddot{y}_1 + \left(-\tfrac{1}{2}L\sin(\theta_1)\right)\ddot{\theta}_1 - \left(\tfrac{1}{2}L\cos(\theta_1)\right)\dot{\theta}_1^2 = 0$$

$$\ddot{x}_2 + \left(-L\cos(\theta_1)\right)\ddot{\theta}_1 + \left(L\sin(\theta_1)\right)\dot{\theta}_1^2 + \left(-\tfrac{1}{2}L\cos(\theta_2)\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}L\sin(\theta_2)\right)\dot{\theta}_2^2 = 0$$

$$\ddot{y}_2 + \left(-L\sin(\theta_1)\right)\ddot{\theta}_1 - \left(L\cos(\theta_1)\right)\dot{\theta}_1^2 + \left(-\tfrac{1}{2}L\sin(\theta_2)\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}L\cos(\theta_2)\right)\dot{\theta}_2^2 = 0$$

$$\ddot{\theta}_1 + \ddot{\theta}_2 = 0$$