Example #26 – Intermediate Dynamics: (Natural Frequencies, Mode Shapes)

Given:
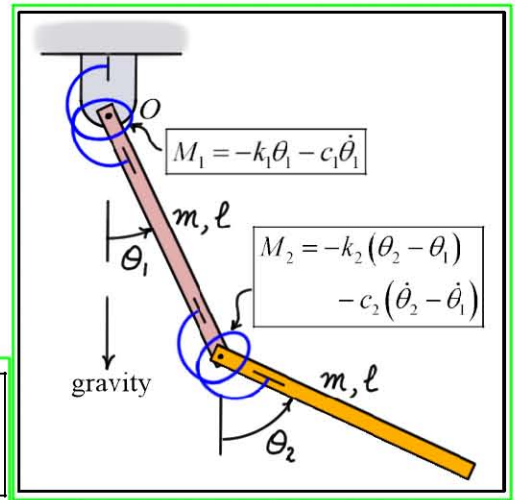
Linearized equations of motion of the double pendulum with adjoining springs and no damping about $\boxed{\theta_1 = \theta_2 = 0}$.

$$[M]\begin{Bmatrix} \Delta\ddot{\theta}_1 \\ \Delta\ddot{\theta}_2 \end{Bmatrix} + [K]\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$M_1 = -k_1\theta_1 - c_1\dot{\theta}_1$

$M_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$

where

$$[M] = \begin{bmatrix} \frac{4}{3}m\ell^2 & \frac{1}{2}m\ell^2 \\ \frac{1}{2}m\ell^2 & \frac{1}{3}m\ell^2 \end{bmatrix} \qquad [K] = \begin{bmatrix} k_1 + k_2 + \frac{3}{2}mg\ell & -k_2 \\ -k_2 & k_2 + \frac{1}{2}mg\ell \end{bmatrix}$$

$\boxed{mg = 5 \text{ (lb)}}$ $\boxed{\ell = 2 \text{ (ft)}}$ $\boxed{k_1 = 1000 \text{ (ft-lb/rad)}}$ $\boxed{k_2 = 300 \text{ (ft-lb/rad)}}$

Find:

Natural frequencies and **mode shapes** of the system.

Solution:

$$[M] = \begin{bmatrix} 0.828157 & 0.310559 \\ 0.310559 & 0.207039 \end{bmatrix} \qquad [K] = \begin{bmatrix} 1315 & -300 \\ -300 & 305 \end{bmatrix}$$

$$\det([K] - \lambda[M]) = \begin{bmatrix} (1315 - 0.828157\lambda) & -(300 + 0.310559\lambda) \\ -(300 + 0.310559\lambda) & (305 - 0.207039\lambda) \end{bmatrix}$$

$$= (0.075014)\lambda^2 - (711.18)\lambda + 311075$$

Characteristic equation: $\boxed{(0.075014)\lambda^2 - (711.18)\lambda + 311075 = 0}$ Roots: $\lambda_{1,2} = \begin{Bmatrix} 459.697 \\ 9020.93 \end{Bmatrix}$

Natural Frequencies: $\omega_{1,2} = \sqrt{\lambda_{1,2}} = \begin{Bmatrix} 21.4405 \text{ (r/s)} \\ 94.9786 \text{ (r/s)} \end{Bmatrix}$ $f_{1,2} = \frac{\omega_{1,2}}{2\pi} = \begin{Bmatrix} 3.4124 \text{ (Hz)} \\ 15.1163 \text{ (Hz)} \end{Bmatrix}$

Mode 1: $\boxed{f_1 = 3.41 \text{ (Hz)}}$

$$([K] - \lambda_1[M])\{u_1\} = \begin{bmatrix} 934.299 & -442.763 \\ -442.763 & 209.825 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} = \{0\}$$

Setting $\boxed{u_{11} = 1}$ and solving either equation for u_{12} gives $\boxed{u_{12} = 2.110}$ $\Rightarrow$ $\{u_1\} = \begin{Bmatrix} 1 \\ 2.110 \end{Bmatrix}$

Mode 2: $\boxed{f_2 = 15.1 \text{ (Hz)}}$

$$([K] - \lambda_2[M])\{u_2\} = \begin{bmatrix} -6155.75 & -3101.53 \\ -3101.53 & -1562.68 \end{bmatrix} \begin{Bmatrix} u_{21} \\ u_{22} \end{Bmatrix} = \{0\}$$

Setting $\boxed{u_{21} = 1}$ and solving either equation for u_{22}: $\Rightarrow$ $\boxed{u_{22} = -1.985}$ $\Rightarrow$ $\{u_2\} = \begin{Bmatrix} 1 \\ -1.985 \end{Bmatrix}$

Mode 1 — in phase motion

Mode 2 — out of phase motion