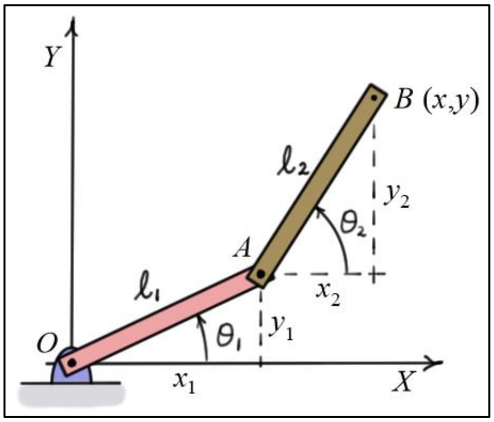
**Elementary Engineering Mathematics**
**Application of Trigonometric Functions in Mechanical Engineering: Part II**




Problem: Find the ***coordinates*** of the endpoint of a two-link planar robot arm.

Given: The ***lengths*** of the links *OA* and *AB* ($\ell_1$ and $\ell_2$) and the ***angles*** $\theta_1$ and $\theta_2$.

Find: The *XY* coordinates of the end-point *B*.

Solution:

The coordinates of *B* can be found by adding the coordinates of *A* ***relative to*** *O* to the coordinates of *B* ***relative to*** *A*. See the diagram.

$\boxed{x = x_1 + x_2 = \ell_1\cos(\theta_1) + \ell_2\cos(\theta_2)}$ and $\boxed{y = y_1 + y_2 = \ell_1\sin(\theta_1) + \ell_2\sin(\theta_2)}$

Example 1:

Given: The ***lengths*** and ***angles*** of a two link planar robot are $\ell_1 = 3\text{ (ft)}$, $\ell_2 = 2\text{ (ft)}$, $\theta_1 = 30\text{ (deg)}$, and $\theta_2 = 60\text{ (deg)}$.

Find: The ***Cartesian coordinates*** *x* and *y* of *B* using a) a calculator, and b) the values listed above for commonly used angles.

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$\boxed{x = \ell_1\cos(\theta_1) + \ell_2\cos(\theta_2) = \big(3\times\cos(30)\big) + \big(2\times\cos(60)\big) \approx 2.5981 + 1 = 3.5981\text{ (ft)}}$$

$$\boxed{y = \ell_1\sin(\theta_1) + \ell_2\sin(\theta_2) = \big(3\times\sin(30)\big) + \big(2\times\sin(60)\big) \approx 1.5 + 1.7321 = 3.2321\text{ (ft)}}$$

b) Using the values for commonly used angles:

$$\boxed{x = \ell_1\cos(\theta_1) + \ell_2\cos(\theta_2) = \left(3\times\tfrac{\sqrt{3}}{2}\right) + \left(2\times\tfrac{1}{2}\right) \approx 2.5981 + 1 = 3.5981\text{ (ft)}}$$

$$\boxed{y = \ell_1\sin(\theta_1) + \ell_2\sin(\theta_2) = \left(3\times\tfrac{1}{2}\right) + \left(2\times\tfrac{\sqrt{3}}{2}\right) \approx 1.5 + 1.7321 = 3.2321\text{ (ft)}}$$

Example 2:

Given: The ***lengths*** and ***angles*** of a two link planar robot are $\ell_1 = 3$ (ft), $\ell_2 = 2$ (ft), $\theta_1 = 30$ (deg), and $\theta_2 = 120$ (deg).

Find: The ***Cartesian coordinates*** *x* and *y* of *B* using a) a calculator, and b) the values listed above for commonly used angles.

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$\boxed{x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \cos(30)\right) + \left(2 \times \cos(120)\right) \approx 2.5981 - 1 = 1.5981 \text{ (ft)}}$$

$$\boxed{y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \sin(30)\right) + \left(2 \times \sin(120)\right) \approx 1.5 + 1.7321 = 3.2321 \text{ (ft)}}$$

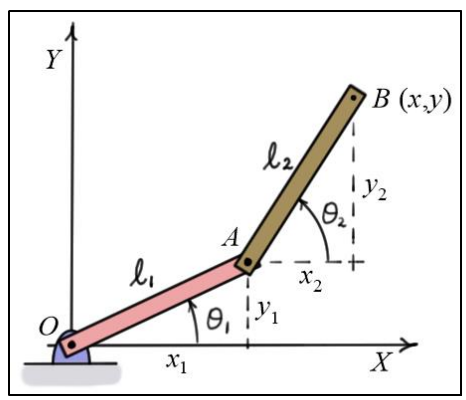
b) Using the values for commonly used angles: Note first that $\boxed{120 = 180 - 60 \text{ (deg)}}$, so

$$\boxed{\cos(120) = -\cos(60) = -\tfrac{1}{2}} \quad \text{and} \quad \boxed{\sin(120) = \sin(60) = \tfrac{\sqrt{3}}{2}}$$

$$\boxed{x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \tfrac{\sqrt{3}}{2}\right) + \left(2 \times (-\tfrac{1}{2})\right) \approx 2.5981 - 1 = 1.5981 \text{ (ft)}}$$

$$\boxed{y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \tfrac{1}{2}\right) + \left(2 \times \tfrac{\sqrt{3}}{2}\right) \approx 1.5 + 1.7321 = 3.2321 \text{ (ft)}}$$

Inverse Problem: Find the ***angles*** of the links of the robot arm given the ***endpoint position***.



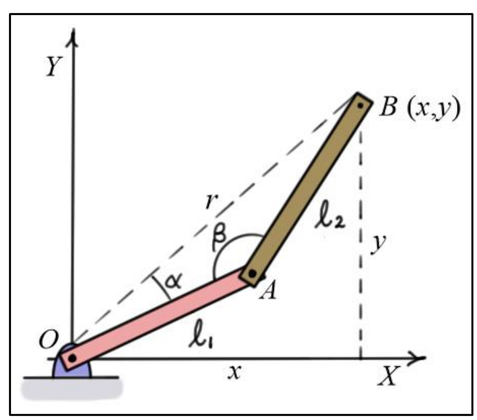

Given: The *XY* ***coordinates*** of the end point *B* and the ***lengths*** of the links *OA* and *AB*.

Find: The link ***angles*** $\theta_1$ and $\theta_2$.

Solution:

First, calculate the length $r$ using the ***Pythagorean Theorem***.

$$\boxed{r = \sqrt{x^2 + y^2}}$$




Then, apply the ***law of cosines*** to triangle *OAB* to find angle $\alpha$.

$$\ell_2^2 = \ell_1^2 + r^2 - 2\ell_1 r \cos(\alpha)$$

or

$$\boxed{\alpha = \cos^{-1}\left(\frac{\ell_1^2 + r^2 - \ell_2^2}{2\ell_1 r}\right)}$$

Finally, apply the ***law of cosines*** again to find angle $\beta$.

$$r^2 = \ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos(\beta) \quad \Rightarrow \quad \boxed{\beta = \cos^{-1}\left(\frac{\ell_1^2 + \ell_2^2 - r^2}{2\ell_1\ell_2}\right)}$$

Finally, the link angles can now be found by noting

a) $\tan(\theta_1 + \alpha) = y/x \quad \Rightarrow \quad \boxed{\theta_1 = \tan^{-1}(y/x) - \alpha}$

b) $\theta_2 - \theta_1 = \pi - \beta \quad \Rightarrow \quad \boxed{\theta_2 = \pi - \beta + \theta_1}$

Example 3:

Given: The *XY* ***coordinates*** of the ***end-point*** *B* and the ***lengths*** of the links *OA* and *AB* are

$x = 1.5$ (ft), $y = 3.5$ (ft), $\ell_1 = 3$ (ft), and $\ell_2 = 2$ (ft).

Find: The link angles $\theta_1$ and $\theta_2$.

Solution:

Following the approach outlined above,

a) $\boxed{r = \sqrt{x^2 + y^2} = \sqrt{1.5^2 + 3.5^2} = 3.8079 \text{ (ft)}}$

b) $2^2 = 3^2 + 3.8079^2 - 2 \times 3 \times 3.8079 \times \cos(\alpha)$

$$\Rightarrow \quad \boxed{\alpha = \cos^{-1}\left(\frac{3^2 + 3.8079^2 - 2^2}{2 \times 3 \times 3.8079}\right) = \begin{cases} 31.41 \text{ (deg)} \\ 0.5481 \text{ (rad)} \end{cases}}$$

c) $r^2 = 3^2 + 2^2 - (2 \times 3 \times 2)\cos(\beta) \quad \Rightarrow \quad \boxed{\beta = \cos^{-1}\left(\frac{3^2 + 2^2 - 3.8079^2}{2 \times 3 \times 2}\right) = \begin{cases} 97.18 \text{ (deg)} \\ 1.6961 \text{ (rad)} \end{cases}}$

d) $\tan(\theta_1 + .5481) = 3.5/1.5 \quad \Rightarrow \quad \boxed{\theta_1 = \tan^{-1}(3.5/1.5) - .5481 = \begin{cases} 35.40 \text{ (deg)} \\ 0.6178 \text{ (rad)} \end{cases}}$

$$\theta_2 - \theta_1 = \pi - \beta \quad \Rightarrow \quad \boxed{\theta_2 = \pi - 1.6961 + 0.6178 = \begin{cases} 118.2 \text{ (deg)} \\ 2.0633 \text{ (rad)} \end{cases}}$$

Check:

We can now use the calculated link angles to check the position of the endpoint. Does it match our required position?

$$\boxed{\begin{aligned} x &= \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) \\ &= \left(3\times\cos(0.6178)\right) + \left(2\times\cos(2.0633)\right) = 2.4455 - 0.9457 = 1.4998 \approx 1.5 \text{ (ft)} \end{aligned}} \checkmark$$

$$\boxed{\begin{aligned} y &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) \\ &= \left(3\times\sin(0.6178)\right) + \left(2\times\sin(2.0633)\right) = 1.7377 + 1.7623 = 3.5 \text{ (ft)} \end{aligned}} \checkmark$$

Note on ***calculator usage***:

When calculating $\sin^{-1}(\theta)$, $\cos^{-1}(\theta)$ and $\tan^{-1}(\theta)$, your calculator will place the results in ***specific quadrants*** as outlined in the table to the right. So, your calculator ***does not always place*** the angle into the correct quadrant.

| Function | Range | Quadrants |
|---|---|---|
| $\sin^{-1}(\theta)$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ | I, IV |
| $\cos^{-1}(\theta)$ | $0 \le \theta \le \pi$ | I, II |
| $\tan^{-1}(\theta)$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ | I, IV |

Note that in the above example, we used the ***law of cosines*** (and hence $\cos^{-1}(\theta)$) to calculate the angles of the triangle *OAB*, and our calculator gave angles in the range $0 \le \theta \le \pi$. What if we had used the ***law of sines*** to calculate the angle $\beta$?

Law of Sines:

$$\frac{\sin(\beta)}{r} = \frac{\sin(\alpha)}{\ell_2}$$

$$\Rightarrow \boxed{\beta = \sin^{-1}\left(r\sin(\alpha)/\ell_2\right) = \sin^{-1}\left(3.8079\times\sin(0.5481)/2\right) = \begin{cases} 82.79 \text{ (deg)} \\ 1.4449 \text{ (rad)} \end{cases}}$$

Note this is ***not*** the correct result. As we know from our work above, the correct result is in the ***second quadrant***. So, $\beta = \pi - 1.4449 = 1.6967 \text{ (rad)}$. This is very close to the result found above.

## Elbow-down and Elbow-up Positions

Note that the above answers could be interpreted in ***two ways*** – the ***elbow-down*** or ***elbow-up*** positions as illustrated in the following diagrams. The ***numerical results*** are the ***same*** but with ***different physical interpretations***. Mathematical results must always be ***interpreted*** with the physical system in mind.