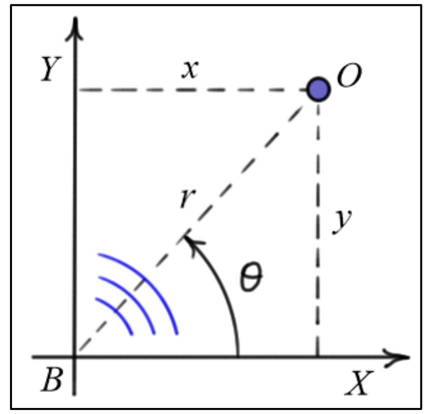
Elementary Engineering Mathematics

Application of Trigonometric Functions in Mechanical Engineering: Part I



Given: The ***position*** of an object O is to be found relative to the base B. The distance r and the angle θ were found using radar.

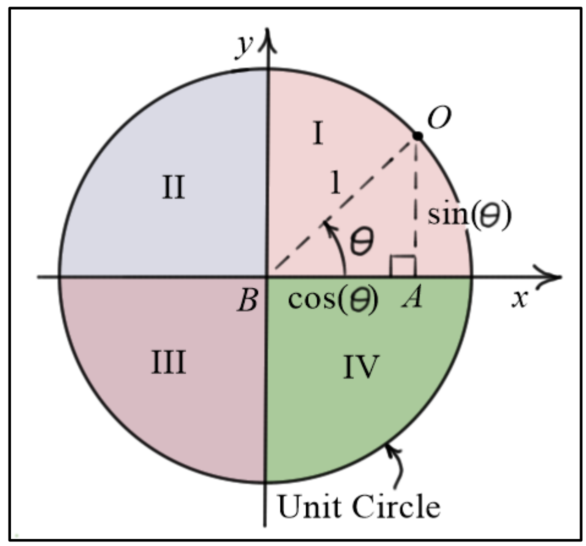
Find: Find the X and Y coordinates of the object O.

Solution: The distances x and y are related to r and θ by the ***trigonometric functions*** defined for a ***right triangle***. In particular,

$$\boxed{\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}} \qquad \boxed{\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}}$$

Note: The distance r and angle θ are called the ***polar coordinates*** of O, and the distances x and y are called the ***Cartesian coordinates*** of O.

Sine and Cosine Functions:



These trigonometric functions can be defined based on lengths found within the ***unit circle*** (a circle of radius one). The triangle ABO is a right triangle with hypotenuse length $r=1$ and sides of length $x=\cos(\theta)$ and $y=\sin(\theta)$. Using the ***Pythagorean theorem***, we see that the $\sin(\theta)$ and $\cos(\theta)$ are related by the following expression.

$$\boxed{\sin^2(\theta)+\cos^2(\theta)=1} \qquad (2)$$

Note also that due to the directions of the X and Y axes, these functions ***change sign*** as θ is varied from $0 \rightarrow \pm 360$ degrees (or $0 \rightarrow \pm 2\pi$ radians). Table 1 summarizes how the $\sin(\theta)$ and $\cos(\theta)$ functions change as θ is varied through each of the ***four quadrants*** (labeled as I, II, III, and IV in the diagram). Table 2 provides the values for some commonly used angles. A more detailed plot of the functions is shown below in Figure 1.

Quadrant	I	II	III	IV
$\sin(\theta)$	"$+$" $(0 \to +1)$	"$+$" $(+1 \to 0)$	"$-$" $(0 \to -1)$	"$-$" $(-1 \to 0)$
$\cos(\theta)$	"$+$" $(+1 \to 0)$	"$-$" $(0 \to -1)$	"$-$" $(-1 \to 0)$	"$+$" $(0 \to +1)$

Table 1. Variation of Sine and Cosine Functions Through the Quadrants

Angle, θ	0	30 (deg) $(\pi / 6 \text{ (rad)})$	45 (deg) $(\pi / 4 \text{ (rad)})$	60 (deg) $(\pi / 3 \text{ (rad)})$	90 (deg) $(\pi / 2 \text{ (rad)})$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Table 2. Values for Commonly Used Angles

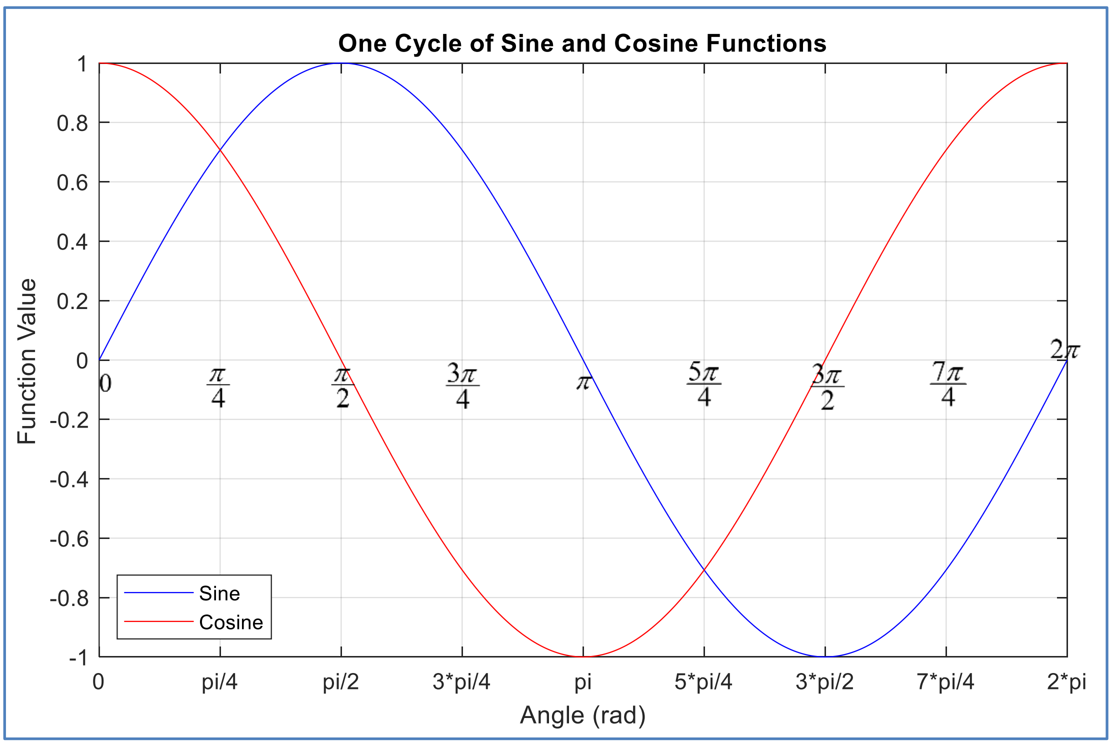


Figure 1. Plot of Sine and Cosine Functions

Tangent Function:

Returning to object O in the first diagram, the distances x and y can be ***related directly*** using the ***tangent*** function

$$r = \frac{x}{\cos(\theta)} = \frac{y}{\sin(\theta)} \quad \Rightarrow \quad \boxed{\frac{y}{x} = \frac{\text{opp}}{\text{adj}} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)} \quad \text{or} \quad \boxed{y = x\tan(\theta)} \qquad (3)$$

Because the $\tan(\theta)$ is a ***ratio*** of $\sin(\theta)$ and $\cos(\theta)$, the ***sign*** of the tangent function is again determined by the ***quadrant*** of the angle. The results are summarized in Table 3. Note that $\tan(\theta)$ is ***undefined*** when $\cos(\theta) = 0$, that is when $\theta = \begin{cases} \pi/2 \text{ (rad) (90 deg)} \\ 3\pi/2 \text{ (rad) (270 deg)} \end{cases}$

Quadrant	I	II	III	IV
$\sin(\theta)$	"+" $(0 \to +1)$	"+" $(+1 \to 0)$	"−" $(0 \to -1)$	"−" $(-1 \to 0)$
$\cos(\theta)$	"+" $(+1 \to 0)$	"−" $(0 \to -1)$	"−" $(-1 \to 0)$	"+" $(0 \to +1)$
$\tan(\theta)$	"+" $(0 \to +\infty)$	"−" $(-\infty \to 0)$	"+" $(0 \to +\infty)$	"−" $(-\infty \to 0)$

Table 3. Variation of Sine, Cosine and Tangent Functions Through the Quadrants

Other Trigonometric Functions:

It is also common to define the reciprocals of the sine, cosine, and tangent functions. These are the ***cosecant***, ***secant***, and ***cotangent*** functions, respectively.

$$\boxed{\csc(\theta) = \frac{1}{\sin(\theta)}} \quad \boxed{\sec(\theta) = \frac{1}{\cos(\theta)}} \quad \boxed{\cot(\theta) = \frac{1}{\tan(\theta)}} \qquad (4)$$

Example 1:

Given: The polar coordinates of an object O are $r = 3500$ (ft) and $\theta = 150$ (deg).

Find: The Cartesian coordinates x and y of O using a) a calculator, and b) the values listed above for commonly used angles

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$\boxed{x = r\cos(\theta) = 3500 \times \cos(150) \approx -3031 \text{ (ft)}}$$

$$\boxed{y = r\sin(\theta) = 3500 \times \sin(150) = 1750 \text{ (ft)}}$$

b) Using the values for commonly used angles: Note first that $\boxed{150 = 180 - 30 \text{ (deg)}}$, so

$$\boxed{\sin(150) = \sin(30) = \tfrac{1}{2}} \quad \text{and} \quad \boxed{\cos(150) = -\cos(30) = -\tfrac{\sqrt{3}}{2}}$$

$$\Rightarrow \boxed{x = r\cos(\theta) = 3500 \times \frac{-\sqrt{3}}{2} \approx -3031 \text{ (ft)}} \text{ and } \boxed{y = r\sin(\theta) = 3500 \times \tfrac{1}{2} = 1750 \text{ (ft)}}$$

<u>Inverse Trigonometric Functions:</u>

Given values for the distances *x*, *y*, and/or *r*, the angle θ can be found using ***inverse trigonometric functions*** as follows:

$$\boxed{\theta = \sin^{-1}(y/r) = \cos^{-1}(x/r) = \tan^{-1}(y/x)}$$

<u>Example 2:</u>

<u>Given</u>: The Cartesian coordinates of an object *O* are $x = -3250$ (ft) and $y = 1250$ (ft).

<u>Find</u>: a) the distance *r*, and b) the angle θ.

<u>Solution</u>:

a) Distance *r* is found by using the Pythagorean theorem:

$$\boxed{r = \sqrt{(-3250)^2 + 1250^2} \approx 3482 \text{ (ft)}}$$

b) The angle can be found using the inverse tangent function, being careful to ***identify*** the ***correct quadrant*** based on the ***algebraic signs*** of *x* and *y*.

$$\boxed{\theta = \tan^{-1}\left(\frac{+1250}{-3250}\right) \approx 159 \text{ (deg)}} \quad \text{or} \quad \boxed{\theta = \tan^{-1}\left(\frac{+1250}{-3250}\right) \approx 2.77 \text{ (rad)}}$$

Your ***calculator*** will probably always given you an angle that is between $-\pi/2$ and $-\pi/2$ radians, so you will have to ***adjust*** the result by adding π (rad) or 180 (deg). Some may have an "atan2" function that determines the quadrant by individually considering the sign of the numerator and denominator of the ratio.