Elementary Engineering Mathematics
Application of Complex Numbers in Electric Circuits

Impedance in AC (Alternating Current) Circuits

In AC circuits, the steady-state voltages and currents are nearly ***sinusoidal***. They alternate at some ***frequency*** ω (rad/sec) and have both ***magnitude*** and ***phase***. We can analyze these signals using ***complex numbers*** and a ***complex form*** of ***Ohm's law***. This form of Ohm's law relates the ***sinusoidal signals*** to the ***impedances*** of the circuit elements. The ***impedances*** are expressed as ***complex numbers*** and are measured in ***ohms*** (Ω). Impedance in an AC circuit is analogous to resistance in a DC (Direct Current) circuit.

Resistor

The impedance of a resistor is its resistance R, $\boxed{Z_R = R + j0 = Re^{j(0)} = R\angle 0^{\text{o}}}$. The impedance of a resistor is the same at all frequencies (ω).

Capacitor

The impedance of a capacitor is $\boxed{Z_C = -j/\omega C = \left(1/\omega C\right)e^{j(-90^{\text{o}})} = \left(1/\omega C\right)\angle\left(-90^{\text{o}}\right)}$. Here, ω is the ***frequency*** of the signals in (rad/s) and C is the ***capacitance*** measured in ***farads*** (f). Often C is provided in ***micro-farads*** (μf). A micro-farad is 10^{-6} farads.

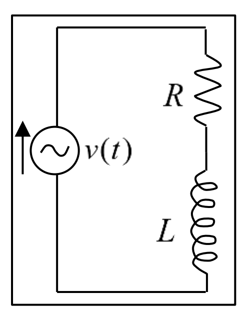
Inductor

The impedance of an inductor is $\boxed{Z_L = j\omega L = \left(\omega L\right)e^{j(90^{\text{o}})} = \left(\omega L\right)\angle(90^{\text{o}})}$. Here, L is the ***inductance*** measured in ***henrys*** (h). Often L is provided in ***millihenrys*** (mh). A millihenry is 10^{-3} henrys. As before, ω is the ***frequency*** of the signals in (rad/s).

Example #1

Given:

A voltage $v(t) = 110\cos(120\pi t)$ is applied to an *RL* series circuit with $R = 100$ (ohms) and $L = 100$ (mh). The impedance of the circuit is $\boxed{Z = Z_R + Z_L}$ and the frequency is $\boxed{\omega = 120\pi \text{ (rad/s)}}$.



Find: Find the complex impedance Z.

Solution:

$$Z = Z_R + Z_L = 100 + j(120\pi)(0.1) \quad \Rightarrow \boxed{Z \approx 100 + j37.7 \text{ (ohms)}}$$

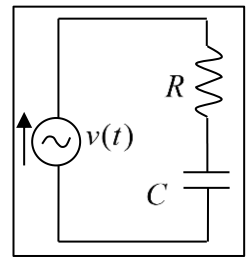
or

$$\boxed{Z \approx 106.9 e^{j(20.7^{\circ})} \approx 106.9\angle(20.7^{\circ}) \text{ (ohms)}}$$

Example #2

Given:

A voltage $v(t) = 110\cos(120\pi t)$ is applied to an *RC* series circuit with $R = 100$ (ohms) and $C = 20$ (μf). The impedance of the circuit is $\boxed{Z = Z_R + Z_C}$ and the frequency is $\boxed{\omega = 120\pi \text{ (rad/s)}}$.



Find: Find the complex impedance Z.

Solution:

$$Z = Z_R + Z_C = 100 - j/(120\pi)(20\times 10^{-6}) \quad \Rightarrow \boxed{Z \approx 100 - j132.6 \text{ (ohms)}}$$

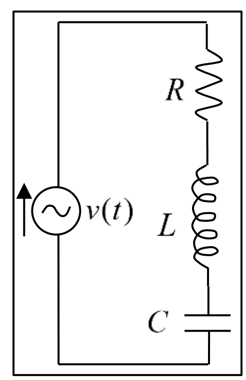
or

$$\boxed{Z \approx 166.1 e^{j(-52.98^{\circ})} \approx 166.1\angle(-52.98^{\circ}) \text{ (ohms)}}$$

Example #3

Given:

A voltage $v(t) = 110\cos(120\pi t)$ is applied to an *RLC* series circuit with $R = 100$ (ohms), $L = 100$ (mh), and $C = 20$ (μf). The impedance of the circuit is $\boxed{Z = Z_R + Z_L + Z_C}$ and the frequency is $\boxed{\omega = 120\pi \text{ (rad/s)}}$.



Find: Find the complex impedance Z.

Solution: (using the results from the examples above)

$$Z = Z_R + Z_L + Z_C \approx 100 + j(37.7 - 132.6) \quad \boxed{Z \approx 100 - j94.9 \text{ (ohms)}}$$

or

$$\boxed{Z \approx 137.9 e^{j(-43.5^{\circ})} \approx 137.9\angle(-43.5^{\circ}) \text{ (ohms)}}$$

Note: The ***impedance*** of ***capacitors*** and ***inductors*** is a ***function of frequency***, whereas the impedance of a resistor is the same at all frequencies.

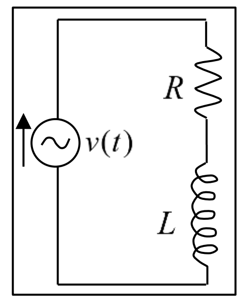
Complex Form of Ohm's Law

To find the currents in the above circuits, we use a complex form of Ohm's law which states that the ***voltage drop across an impedance*** is equal to the product of the complex current and complex impedance. That is, $\boxed{V = I\,Z}$. Rewriting this equation, we can find the current $\boxed{I = V/Z}$.

Example #4

Given:

A voltage $v(t) = 110\cos(120\pi t)$ is applied to an *RL* series circuit with $R = 100$ (ohms) and $L = 100$ (mh). The impedance of the circuit is $\boxed{Z = Z_R + Z_L}$ and the frequency is $\boxed{\omega = 120\pi \text{ (rad/s)}}$.



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, the impedance is $\boxed{Z = Z_R + Z_L \approx 100 + j37.7 \text{ (ohms)}}$

or

$$\boxed{Z \approx 106.9e^{j(20.7^\circ)} \approx 106.9\angle(20.7^\circ) \text{ (ohms)}}$$

The given voltage $v(t)$ can be written in complex form as $\boxed{V = 110e^{j(0^\circ)} = 110\angle(0^\circ)}$.

So, the complex current is

$$\boxed{I = V/Z \approx 110\,e^{j(0^\circ)}\big/106.9e^{j(20.7^\circ)} \approx 1.029e^{j(-20.7^\circ)} \text{ (amps)}}$$

and

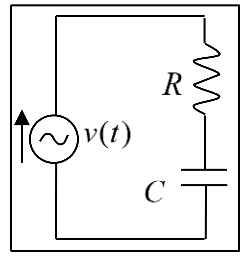
$\boxed{i(t) \approx 1.029\cos(120\pi t - 0.3605) \text{ (amps)}}$ (argument of cosine function ~ radians)

Note: We use only the ***phase angle*** of $v(t)$ and not "$\omega\,t$" when we write the complex form. We must include the "$\omega\,t$" when we write $i(t)$.

Example #5

Given:

A voltage $v(t) = 110\cos(120\pi t)$ is applied to an *RC* series circuit with $R = 100$ (ohms) and $C = 20$ (μf). The impedance of the circuit is $\boxed{Z = Z_R + Z_C}$ and the frequency is $\boxed{\omega = 120\pi \text{ (rad/s)}}$.



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, the impedance is $\boxed{Z = Z_R + Z_C \approx 100 - j132.6 \text{ (ohms)}}$

or

$$\boxed{Z \approx 166.1e^{j(-52.98^\circ)} \approx 166.1\angle(-52.98^\circ) \text{ (ohms)}}$$

So, the complex current is

$$\boxed{I = V/Z \approx 110e^{j(0^\circ)} \big/ 166.1e^{j(-52.98^\circ)} \approx 0.6623e^{j(52.98^\circ)} \text{ (amps)}}$$

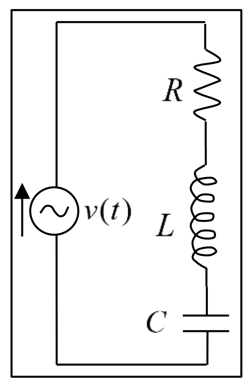
and

$\boxed{i(t) \approx 0.6623\cos(120\pi t + 0.9246) \text{ (amps)}}$ (argument of cosine function ~ radians)

Example #6

Given:

A voltage $v(t) = 110\cos(120\pi t)$ is applied to an *RLC* series circuit with $R = 100$ (ohms), $L = 100$ (mh), and $C = 20$ (μf). The impedance of the circuit is $\boxed{Z = Z_R + Z_L + Z_C}$ and the frequency is $\boxed{\omega = 120\pi \text{ (rad/s)}}$.



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, we have $\boxed{Z = Z_R + Z_L + Z_C \approx 100 - j94.9 \text{ (ohms)}}$

or

$$\boxed{Z \approx 137.9e^{j(-43.5^\circ)} \approx 137.9\angle(-43.5^\circ) \text{ (ohms)}}$$

So, the complex current is

$$\boxed{I = V/Z \approx 110e^{j(0^\circ)} \big/ 137.9e^{j(-43.5^\circ)} \approx 0.7977e^{j(43.5^\circ)} \text{ (amps)}}$$

and

$$\boxed{i(t) = 0.7977\cos(120\pi t + 0.7592) \text{ (amps)}} \quad \text{(argument of cosine function} \sim \text{radians)}$$

The graph below shows a plot of three cycles of the voltage $v(t)$ and current $i(t)$ for the *RLC* circuit.