Elementary Engineering Mathematics
Applications of Systems of Linear, Algebraic Equations in ME 2560

Commonly Used Methods

Consider a system of n linear algebraic equations in n unknown variables $x_1, x_2, \cdots, x_n$. The equations can be written ***algebraically*** or in ***matrix form*** as

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}} \quad \text{or} \quad \boxed{[A]\{x\} = \{b\}} \qquad (1)$$

Here, $[A]$ is an $n \times n$ matrix of coefficients, $\{x\}$ is the $n \times 1$ vector of unknowns, and $\{b\}$ is an $n \times 1$ vector of known values. That is,

$$[A]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad \{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

Four commonly used methods used to solve these equations, that is, find the vector of unknowns $\{x\}$ that satisfies the equations are

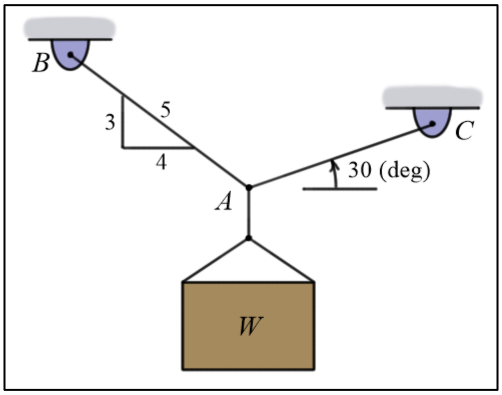
- Graphical Method
- Substitution (or Gaussian Elimination)
- Cramer's Rule
- Matrix Inversion

What if we cannot find a solution?

In general, Eq. (1) ***may*** or ***may not*** have a solution. If we find that no solution exists, then from an engineering prospective, there are two possibilities. We have either failed to describe the physical problem correctly, or the physical problem itself has no solution. It is up to us as engineers to determine which is the case. As we apply the above methods, we will see the conditions under which no solution exists.

Example: Elementary Statics

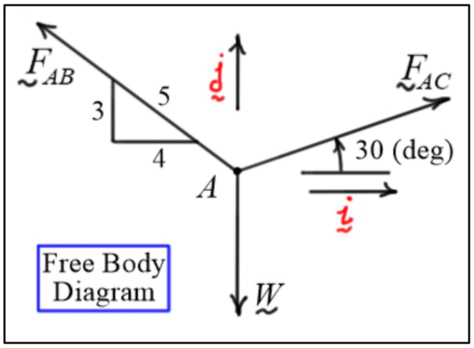
Given: Weight $W = 500$ (lb).



Find: The forces in the cables AB and AC required to hold the weight in ***equilibrium***.

Solution:

For point A to be in ***equilibrium***, the ***sum*** of all ***forces*** acting there must be ***zero***. Using the free body diagram,



$$\boxed{\underset{\sim}{F}_{AB} + \underset{\sim}{F}_{AC} + \underset{\sim}{W} = \underset{\sim}{0}} \qquad (2)$$

where

$$\underset{\sim}{F}_{AB} = F_{AB}\left(-\tfrac{4}{5}\underset{\sim}{i} + \tfrac{3}{5}\underset{\sim}{j}\right) \text{ (lb)}$$

$$\underset{\sim}{F}_{AC} = F_{AC}\left(\cos(30)\,\underset{\sim}{i} + \sin(30)\,\underset{\sim}{j}\right) = F_{AC}\left(\tfrac{\sqrt{3}}{2}\underset{\sim}{i} + \tfrac{1}{2}\underset{\sim}{j}\right) \text{ (lb)}$$

$$\underset{\sim}{W} = -500\,\underset{\sim}{j} \text{(lb)}$$

Substituting into Eq. (2):

$$F_{AB}\left(-\tfrac{4}{5}\underset{\sim}{i} + \tfrac{3}{5}\underset{\sim}{j}\right) + F_{AC}\left(\tfrac{\sqrt{3}}{2}\underset{\sim}{i} + \tfrac{1}{2}\underset{\sim}{j}\right) + \left(-500\,\underset{\sim}{j}\right) = \underset{\sim}{0}$$

$$\left(-\tfrac{4}{5}F_{AB} + \tfrac{\sqrt{3}}{2}F_{AC}\right)\underset{\sim}{i} + \left(\tfrac{3}{5}F_{AB} + \tfrac{1}{2}F_{AC} - 500\right)\underset{\sim}{j} = \underset{\sim}{0}$$

This last equation can be written as two linear, algebraic equations for the unknowns F_{AB} and F_{AC}. To find these forces, we must solve the equations simultaneously.

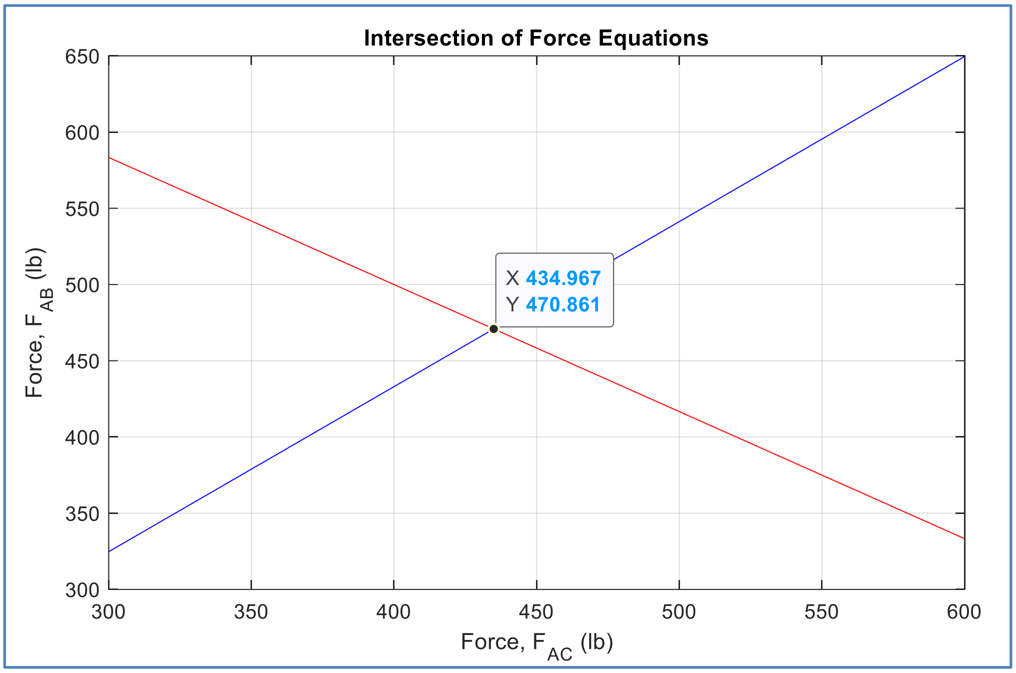
$$\boxed{\begin{aligned} -\tfrac{4}{5}F_{AB} + \tfrac{\sqrt{3}}{2}F_{AC} &= 0 \\ \tfrac{3}{5}F_{AB} + \tfrac{1}{2}F_{AC} &= 500 \end{aligned}}$$

Graphical Method: (for equations with two unknowns)

In this method, we think of each equation as the equation of a line and look for the intersection of the two lines. That point would be a solution to both equations.

$$-\tfrac{4}{5}F_{AB} + \tfrac{\sqrt{3}}{2}F_{AC} = 0 \quad \Rightarrow \quad \boxed{F_{AB} = \tfrac{5\sqrt{3}}{8}F_{AC} \approx 1.0825 F_{AC}}$$

$$\tfrac{3}{5}F_{AB} + \tfrac{1}{2}F_{AC} = 500 \quad \Rightarrow \quad \boxed{F_{AB} = \tfrac{2500}{3} - \tfrac{5}{6}F_{AC} = 833.\overline{3} - \tfrac{5}{6}F_{AC}}$$



From the graph, we see that if $F_{AC} \approx 434.97$ (lb) and $F_{AC} \approx 470.86$ (lb) ***both equations*** are ***satisfied***. Also, it is easy to visualize that ***no solution*** would exist if the ***two lines*** are ***parallel***. In that case, there would be ***no intersection point***.

Substitution (Gaussian Elimination):

In the method of substitution (or Gaussian Elimination), we solve the first equation for one of the variables, say the first one, and substitute that into the second equation. Then, use the resulting equation to solve for the second variable. We can then substitute this result into the first equation and solve for the first variable.

$$-\tfrac{4}{5}F_{AB} + \tfrac{\sqrt{3}}{2}F_{AC} = 0 \quad \Rightarrow \quad \boxed{F_{AB} = \tfrac{5\sqrt{3}}{8}F_{AC}}$$

$$\tfrac{3}{5}F_{AB} + \tfrac{1}{2}F_{AC} = 500 \quad \Rightarrow \quad \tfrac{3}{5}\left(\tfrac{5\sqrt{3}}{8}F_{AC}\right) + \tfrac{1}{2}F_{AC} = 500 \quad \Rightarrow \quad \underbrace{\left(\tfrac{3\sqrt{3}}{8} + \tfrac{1}{2}\right)}_{1.1495}F_{AC} = 500$$

or

$$\boxed{F_{AC} = 500/1.1495 \approx 434.97 \text{ (lb)}} \quad \text{and} \quad \boxed{F_{AB} = \tfrac{5\sqrt{3}}{8}F_{AC} \approx 1.0825 \times 434.97 \approx 470.87 \text{ (lb)}}$$

Substitution has the ***advantage*** of not requiring us to plot the functions, but it has the ***disadvantage*** of not understanding the condition under which no solution exists. We just immerse ourselves in the details of the algebraic solution.

Cramer's Rule:

To use Cramer's Rule, we first express the equations in ***matrix form***, and then calculate the unknowns one at a time. An illustration of how to apply Cramer's Rule to a set of three equations follows. Extension of the rule to larger sets of equations should be obvious.

$$[A]\{x\}=\{b\} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

The solution for each of the x_i $(i=1,2,3)$ are calculated using ***determinants*** as follows

$$x_1 = \frac{\det\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}{\det[A]} \qquad x_2 = \frac{\det\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}}{\det[A]} \qquad x_3 = \frac{\det\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}}{\det[A]}$$

Clearly, solutions can be found only if $\boxed{\det[A]\neq 0}$.

We now return to our example from Elementary Statics. We first write our simultaneous force equations in matrix form, and then we solve for the two unknowns, individually.

$$\begin{aligned} -\tfrac{4}{5}F_{AB}+\tfrac{\sqrt{3}}{2}F_{AC} &= 0 \\ \tfrac{3}{5}F_{AB}+\tfrac{1}{2}F_{AC} &= 500 \end{aligned} \quad \Rightarrow \quad [A]\{f\} = \begin{bmatrix} -\frac{4}{5} & \frac{\sqrt{3}}{2} \\ \frac{3}{5} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} F_{AB} \\ F_{AC} \end{Bmatrix} = \{b\} = \begin{Bmatrix} 0 \\ 500 \end{Bmatrix}$$

$$F_{AB} = \frac{\det\begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ 500 & \frac{1}{2} \end{bmatrix}}{\det\begin{bmatrix} -\frac{4}{5} & \frac{\sqrt{3}}{2} \\ \frac{3}{5} & \frac{1}{2} \end{bmatrix}} = \frac{\left((0\times\frac{1}{2})-(\frac{\sqrt{3}}{2}\times 500)\right)}{\left((-\frac{4}{5}\times\frac{1}{2})-(\frac{\sqrt{3}}{2}\times\frac{3}{5})\right)} \approx \frac{-433.013}{-0.9196} \quad \Rightarrow \quad \boxed{F_{AB}\approx 470.86\ \text{(lb)}}$$

$$F_{AC} = \frac{\det\begin{bmatrix} -\frac{4}{5} & 0 \\ \frac{3}{5} & 500 \end{bmatrix}}{\det\begin{bmatrix} -\frac{4}{5} & \frac{\sqrt{3}}{2} \\ \frac{3}{5} & \frac{1}{2} \end{bmatrix}} = \frac{\left((-\frac{4}{5}\times 500)-(0\times\frac{3}{5})\right)}{\left((-\frac{4}{5}\times\frac{1}{2})-(\frac{\sqrt{3}}{2}\times\frac{3}{5})\right)} \approx \frac{-400}{-0.9196} \quad \Rightarrow \quad \boxed{F_{AC}\approx 434.97\ \text{(lb)}}$$

Matrix Inversion:

To use matrix inversion, we again express our equations in ***matrix form***. Then, we multiply both sides of the equation by the inverse of the coefficient matrix.

$$\boxed{[A]\{x\}=\{b\} \quad\Rightarrow\quad [A]^{-1}[A]\{x\}=[A]^{-1}\{b\} \quad\Rightarrow\quad \{x\}=[A]^{-1}\{b\}}$$

Calculating the inverse of a matrix can be quite ***tedious*** for larger matrices. However, we will work only with 2×2 matrices for which it is easy to calculate the inverse. For a 2×2 matrix,

$$[A]=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad\Rightarrow\quad \boxed{[A]^{-1}=\frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{\det[A]}=\frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{\left(a_{11}a_{22}-a_{12}a_{21}\right)}}$$

Clearly, $[A]^{-1}$ exists only if $\boxed{\det[A]\neq 0}$.

We now return again to our example. We write our simultaneous force equations in matrix form, and then solve for the two unknowns.

$$\boxed{[A]\{f\}=\begin{bmatrix} -\frac{4}{5} & \frac{\sqrt{3}}{2} \\ \frac{3}{5} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} F_{AB} \\ F_{AC} \end{Bmatrix}=\{b\}=\begin{Bmatrix} 0 \\ 500 \end{Bmatrix} \quad\Rightarrow\quad [A]^{-1}\approx\frac{1}{-0.9196}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}}$$

$$\boxed{\begin{Bmatrix} F_{AB} \\ F_{AC} \end{Bmatrix}=[A]^{-1}\{b\}\approx\frac{1}{-0.9196}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}\begin{Bmatrix} 0 \\ 500 \end{Bmatrix}\approx\begin{Bmatrix} 470.87 \\ 434.97 \end{Bmatrix}\ \text{(lb)}}$$