Elementary Engineering Mathematics
Exponential and Natural Logarithm Functions

The exponential function is defined as: $\boxed{f(x)=e^x}$ $(e=2.71828\ldots)$. This function is useful for describing ***growth*** or ***decay rates***. We will see in later notes how it is used to describe the motion of a mass-spring-damper system. Figures 1 and 2 show exponential functions having various growth and decay rates. The ***larger*** the exponent (in absolute value), the ***higher*** the growth or decay rate.

As defined, the exponential function is related to the natural logarithm function $\left(\log_e(x)\right)$. In fact, the exponential function and the natural logarithm functions are ***inverses*** of each other. As a result,

$$\boxed{\log_e(e^x)=\ln(e^x)=x} \quad \text{and} \quad \boxed{e^{\log_e(x)}=e^{\ln(x)}=x} \qquad (1)$$

Figure 3 shows a plot of the exponential and natural logarithm functions. They are mirror images about the line $y=x$.

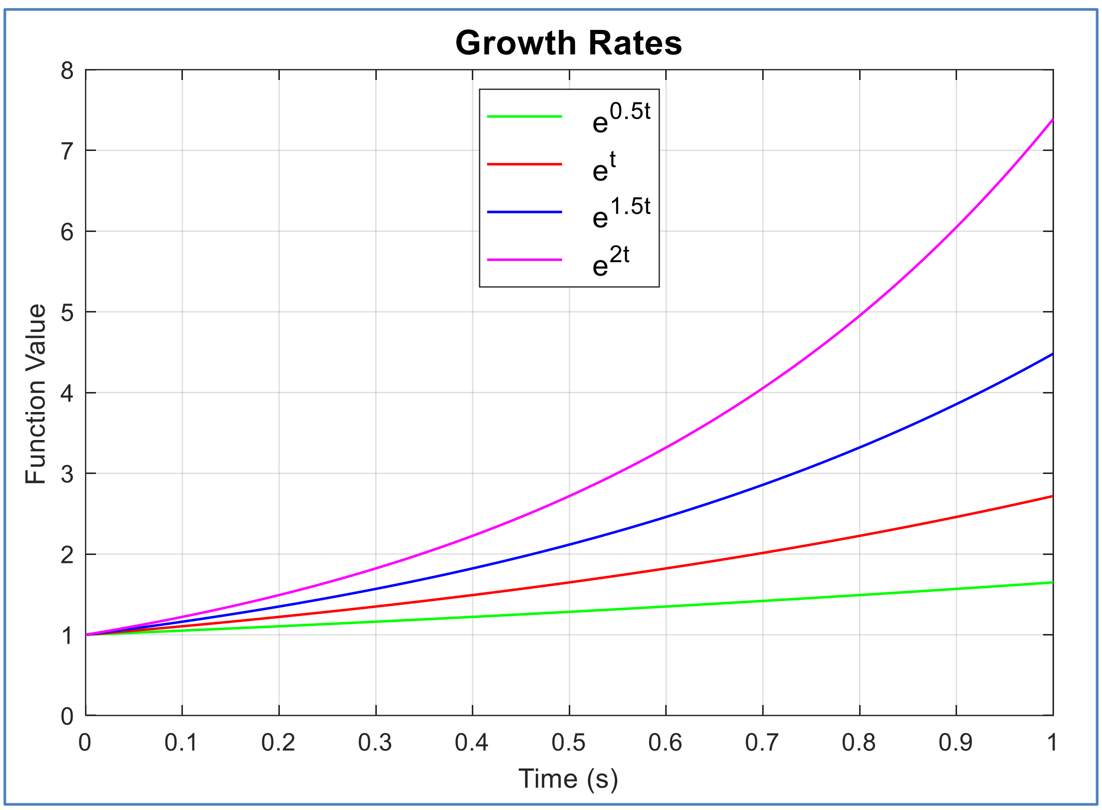


Figure 1. Exponential Growth Rates

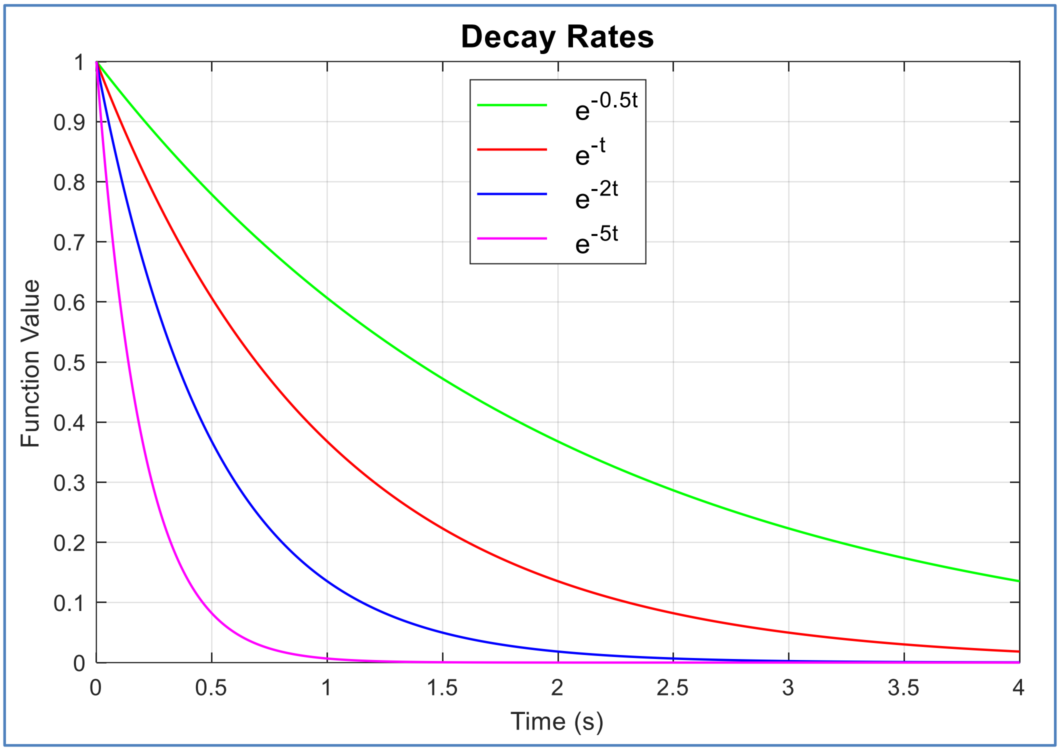


Figure 2. Exponential Decay Rates

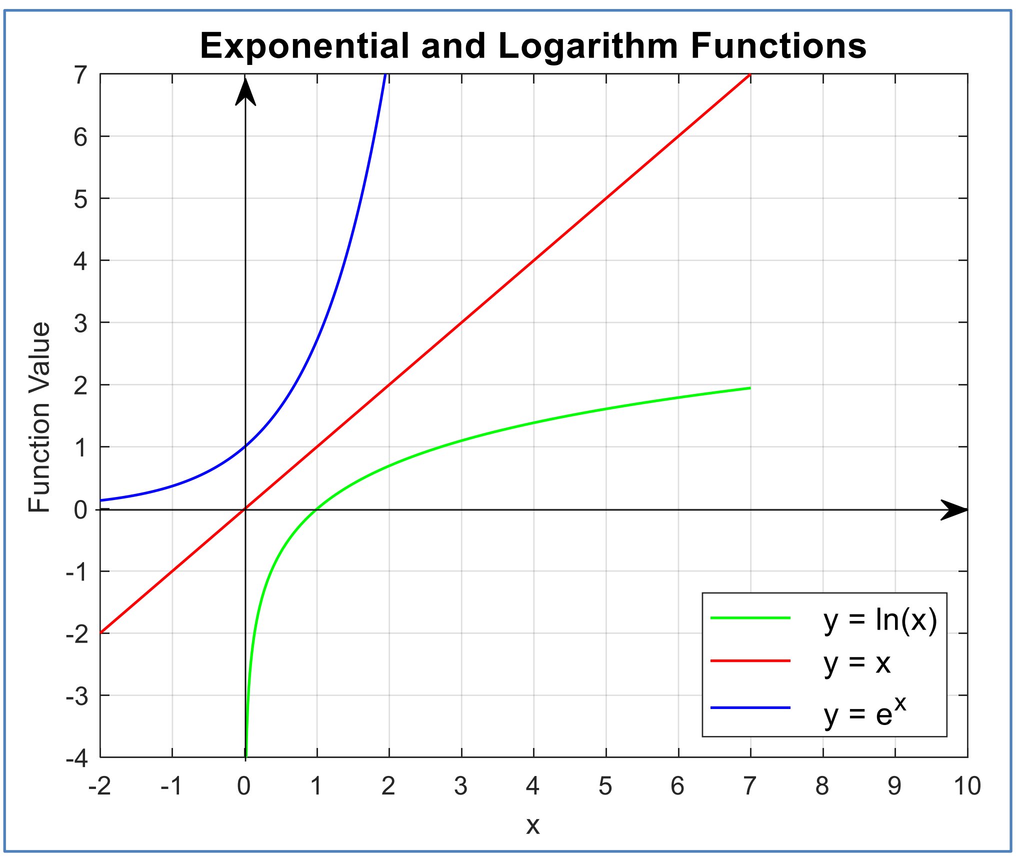
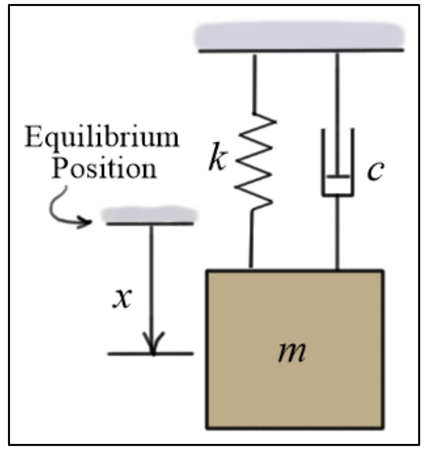


Figure 3. Comparison of Exponential and Logarithm Functions

Example #1: Displacement of a spring-mass-damper system



Given: The response $x(t)$ of the spring-mass-damper system to an initial displacement from its equilibrium position is shown in Figure 4 below. The response is bounded by the two exponential function envelopes. The displacements at six bottom-most positions during roughly a two-second period were measured and are presented in the following table.

Time, *t* (sec)	Displacement, *x* (in)
0.314	3.55
0.631	2.10
0.945	1.24
1.261	0.73
1.576	0.43
1.892	0.26

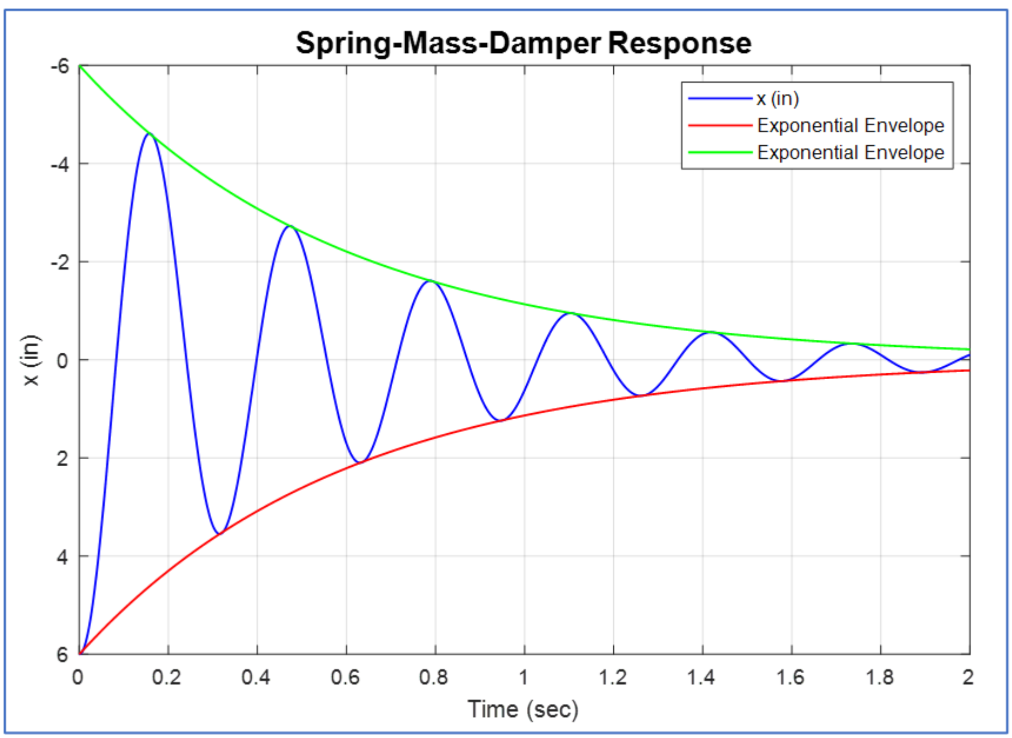


Figure 4. Displacement Response of a Spring-Mass-Damper System

Find: Using the data provided, estimate the decay rate α of the oscillations. Assume the displacement function is bounded by an exponential function $x = Ae^{\alpha t}$.

Solution:

Given that all six points lie on the same exponential function, we can estimate the decay rate by using any pair of data values as follows.

$$\frac{x_2}{x_1} = \frac{\cancel{A}\, e^{\alpha t_2}}{\cancel{A}\, e^{\alpha t_1}} = e^{\alpha (t_2 - t_1)} \quad \Rightarrow \quad \ln\left(\frac{x_2}{x_1}\right) = \ln\left(e^{\alpha (t_2 - t_1)}\right) = \alpha (t_2 - t_1)$$

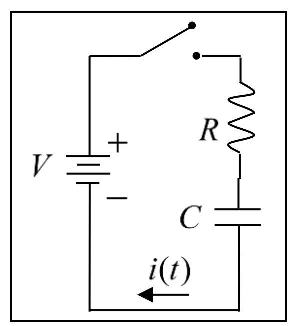
or

$$\boxed{\alpha = \frac{\ln\left(x_2/x_1\right)}{t_2 - t_1} = \frac{\ln(x_2) - \ln(x_1)}{t_2 - t_1}} \qquad (2)$$

Using Eq. (2), we can compute estimates of the decay rate. The six data values provide five estimates of α , and those five values can be averaged to provide a final estimate.

Time, t (sec)	Displacement, x (in)	Decay Rate, α
0.314	3.55	
0.631	2.10	-1.6562
0.945	1.24	-1.6778
1.261	0.73	-1.6767
1.576	0.43	-1.6802
1.892	0.26	-1.5921
	Average	-1.6566

Example #2: Transient response of a direct current RC circuit



Given: For this circuit, $V = 100$ (volts), $R = 500\,(\Omega)$, and $C = 250\,(\mu\text{f})$. When the switch is closed, it can be shown that the resulting current is $\boxed{i(t) = \left[\frac{V}{R}\right] e^{-t/RC}}$.

Find: Calculate the exponential decay rate and plot $i(t)$.

Solution: The decay rate is $\boxed{\alpha = -1/(500)(250 \times 10^{-6}) = -8}$. See the plot below.

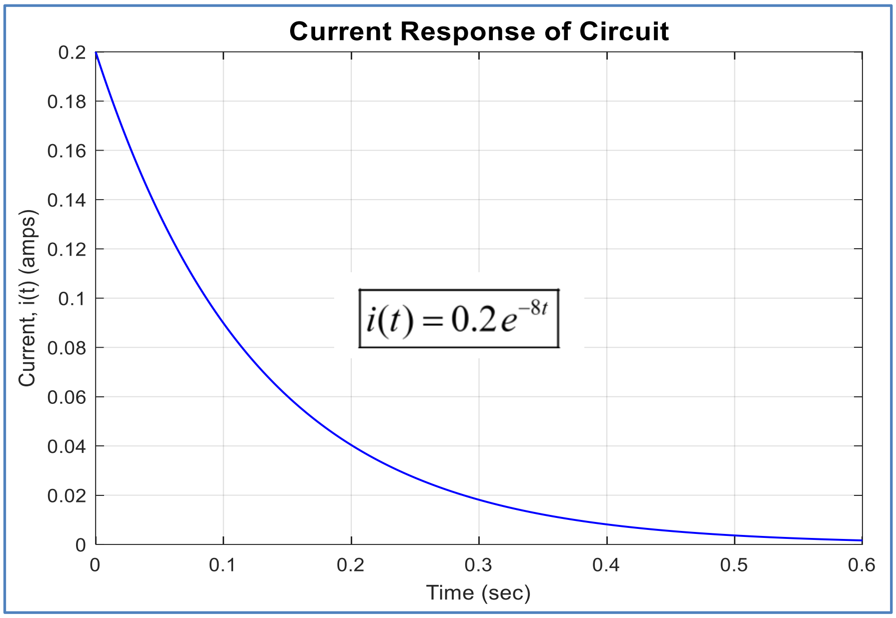
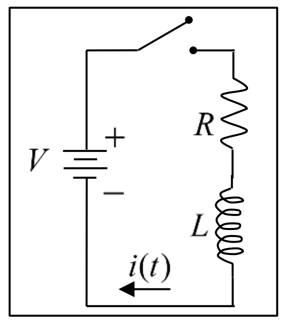


Figure 5. Current Response of Simple RC Circuit



Example #3: Transient response of a direct current RL circuit

Given: For this circuit, $V = 100$ (volts), $R = 100(\Omega)$, and $L = 500(\text{mh})$. When the switch is closed, it can be shown that the resulting current is $\boxed{i(t) = \left[\frac{V}{R}\right]\left(1 - e^{-t(R/L)}\right)}$.

Find: Calculate the exponential decay rate and plot $i(t)$.

Solution: The decay rate is $\boxed{\alpha = -100/(500 \times 10^{-3}) = -200}$.

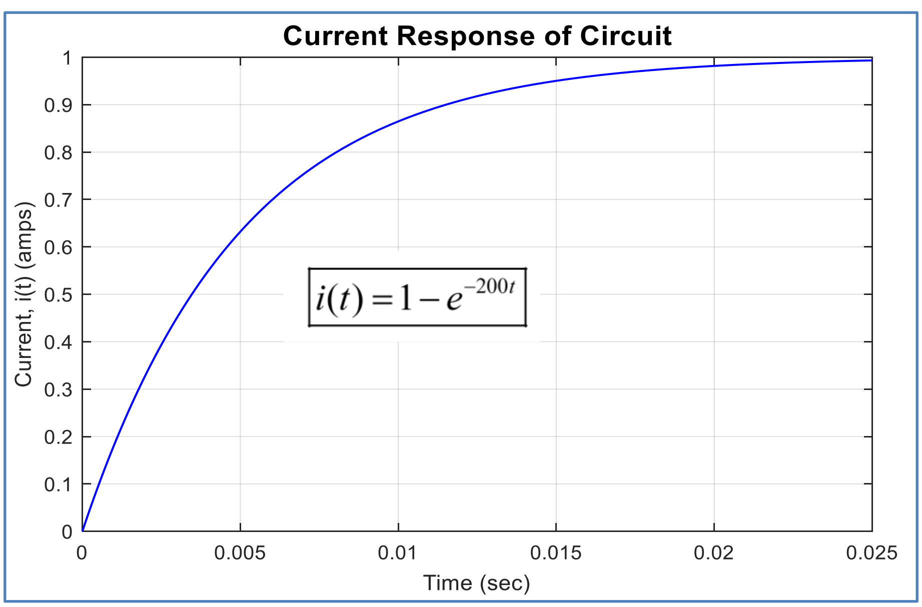


Figure 6. Current Response of Simple RL Circuit