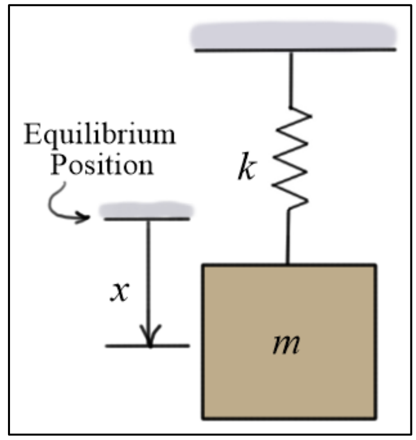
Elementary Engineering Mathematics
Application of Sine, Cosine, and Exponential Functions in Control Systems

Vibration



Vibration refers to oscillatory motion of a body or structure about an ***equilibrium position***. In the case of a simple spring-mass system, the equilibrium position is the ***at-rest position*** of the spring-supported mass. In structural systems, vibrations occur around the at-rest shape of the structure. In most cases, vibrations represent relatively small displacements of the system.

Vibrational motion can be either ***free*** or ***forced***. In the case of ***free motion***, only gravity continues to act on the system as it moves, whereas in ***forced motion***, some other external force or forces continue to excite the system as it moves. ***Free motion*** can be categorized as ***damped*** or ***undamped***. Damped motion eventually decays away to zero while undamped motion continues forever (in theory). Obviously, all systems have some damping, but we often neglect damping as we analyze systems if its effects are small.

Undamped, Free Vibration

Neglecting damping, the ***position*** and ***velocity*** of the mass shown above can be written as

$$\boxed{x(t) = A\sin(\omega t) + B\cos(\omega t)} \quad \text{and} \quad \boxed{v(t) = A\omega\cos(\omega t) - B\omega\sin(\omega t)} \qquad (1)$$

These equations describe the position and velocity of the mass as ***functions of time***. The variable ω (omega) represents the ***frequency*** of the motion in radians per second. This frequency is often referred to as the ***natural frequency*** of the system and is related to the mass and spring stiffness as follows

$$\boxed{\omega = \sqrt{k/m} \text{ (rad/s)}} \qquad (2)$$

The constants A and B are determined from the initial conditions. For example, if the mass has initial displacement $x(0) = x_0$ and initial velocity $v(0) = v_0$, then

$$\boxed{x(0) = x_0 = \left(A\sin(\omega t) + B\cos(\omega t)\right)_{t=0} = B}$$

$$\boxed{v(0) = v_0 = \left(A\omega\cos(\omega t) - B\omega\sin(\omega t)\right)_{t=0} = A\omega}$$

Given the above results, the displacement function is

$$\boxed{x(t)=\tfrac{v_0}{\omega}\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)+x_0\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)} \tag{3}$$

The result in Eq. (3) can be written as a single ***phase-shifted sine wave*** as follows. First, substitute the following equations for A and B into Eq. (3)

$$\boxed{A=\tfrac{v_0}{\omega}=M\cos(\phi)} \quad \text{and} \quad \boxed{B=x_0=M\sin(\phi)} \tag{4}$$

Then use the trigonometric identity $\boxed{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)=\sin(\alpha+\beta)}$ to get

$$\boxed{\begin{aligned} x(t)&=M\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)\cos(\phi)+M\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)\sin(\phi)\\ &=M\left(\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)\cos(\phi)+\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)\sin(\phi)\right)\\ &=M\sin\left(\sqrt{\tfrac{k}{m}}\,t+\phi\right)\end{aligned}} \tag{5}$$

The amplitude M and the phase angle ϕ can be determined from Eq. (4) by noting

$$\boxed{A^2+B^2=M^2\cos^2(\phi)+M^2\sin^2(\phi)=M^2\left(\cos^2(\phi)+\sin^2(\phi)\right)=M^2} \tag{6}$$

$$\boxed{\frac{B}{A}=\frac{M\sin(\phi)}{M\cos(\phi)}=\tan(\phi)} \quad \text{or} \quad \boxed{\phi=\tan^{-1}\left(B/A\right)} \tag{7}$$

The result in Eq. (3) can also be written as a single ***phase-shifted cosine wave***. In this case, it can be shown that

$$\boxed{x(t)=M\cos\left(\sqrt{\tfrac{k}{m}}\,t+\phi\right)} \quad \boxed{M=\sqrt{A^2+B^2}} \quad \boxed{\phi=\tan^{-1}\left(-A/B\right)} \tag{8}$$

Example:

Given: $m=0.5$ (slug), $k=50$ (lb/ft), $x_0=0.25$ (ft), $v_0=5$ (ft/s)

Find: a) $x(t)$ as a sum of sine and cosine functions, b) $x(t)$ as a single sine function with magnitude and phase, c) the time shift of the sine function, d) the time when the mass first reaches its largest displacement, and e) T the period of the oscillation.

Solution:

a) $\omega=\sqrt{k/m}=\sqrt{50/0.5}=\sqrt{100}=10$ (rad/s) $\Rightarrow$ $\boxed{x(t)=\tfrac{5}{10}\sin\left(10\,t\right)+\tfrac{1}{4}\cos\left(10\,t\right)\ \ \text{(ft)}}$

b) $\boxed{M=\sqrt{0.5^2+0.25^2}\approx 0.559 \text{ (ft)}}$ and

$$\boxed{\phi=\tan^{-1}(B/A)=\tan^{-1}\left(\tfrac{1}{4}\big/\tfrac{5}{10}\right)=\tan^{-1}\left(\tfrac{1}{2}\right)\approx\begin{cases}26.565 \text{ (deg)}\\ 0.4636 \text{ (rad)}\end{cases}}$$

$$\Rightarrow \boxed{x(t)\approx 0.559\sin(10\,t+0.4636) \quad \text{(ft)}}$$

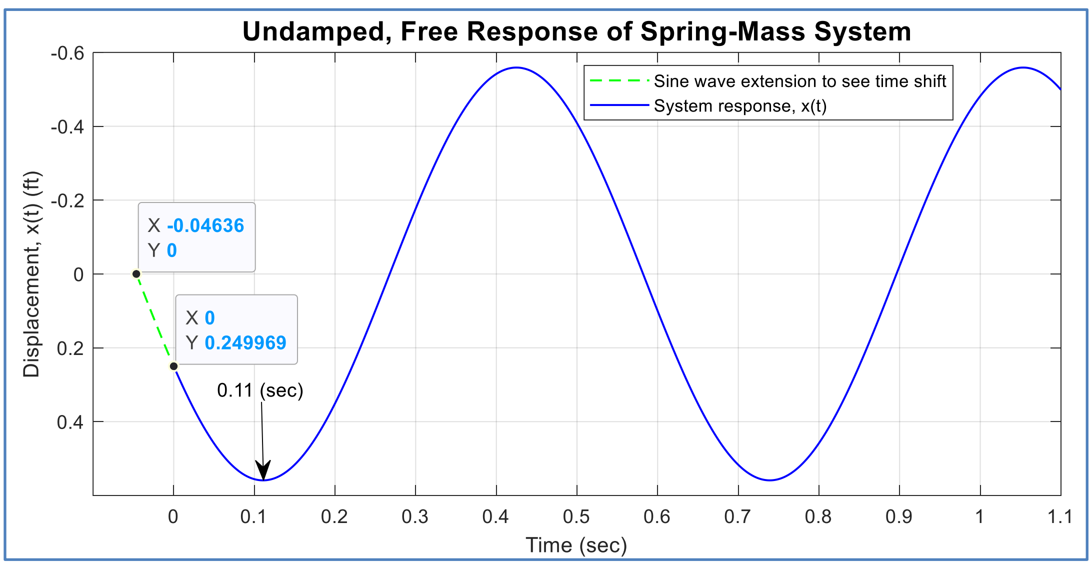
c) To find the time shift, set the argument of the sine function to zero, and solve for the time.

$$10\,t+0.4636=0 \quad\Rightarrow\quad t=-0.4636/10=-0.04636 \quad\Rightarrow\quad \boxed{\text{time shift}=0.04636 \text{ (sec)}}$$

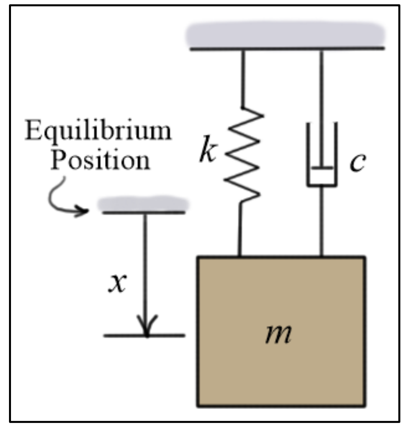
d) The maximum displacement will occur when the ***argument*** of the sine function is an ***odd multiple*** of $\pi/2$. So, set

$$10\,t+0.4636=\pi/2=1.5708 \quad\Rightarrow \boxed{t=(1.5708-0.4636)/10\approx 0.1107\approx 0.11 \text{ (sec)}}$$

e) $\boxed{f=\frac{10}{2\pi}\approx 1.59 \text{ (Hz)}}$ and $\boxed{T\approx\frac{1}{1.5915}\approx 0.628 \text{ (sec)}}$



Damped, Free Vibration



If a ***viscous damper*** is added to the system, then it will no longer oscillate indefinitely. It will eventually come to rest again at the equilibrium position. The character of the response as it returns depends on how much damping is present. If there is not enough damping to eliminate the oscillations, the system is said to be ***under-damped***. If there is enough damping to eliminate the oscillations, the system is said to be ***over-damped***.

To determine which type of response a system will have, we calculate c_c the ***critical damping coefficient***.

$$\boxed{c_c = 2m\sqrt{k/m}} \tag{9}$$

If the damping coefficient c is greater than c_c $(c > c_c)$, the response will be ***over-damped***, and if c is less than c_c $(c < c_c)$, the response will be ***under-damped***. If $c = c_c$, the response is ***critically damped***.

Over-Damped Vibration: $\boxed{c > c_c}$

In this case, the mass ***does not oscillate*** as it returns to the equilibrium position. Instead, the mass moves directly to the its final position, slowing down as it approaches. The position and velocity of the mass can be written as the sum of two exponential functions

$$\boxed{\begin{aligned} x(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ v(t) &= A\lambda_1 e^{\lambda_1 t} + B\lambda_2 e^{\lambda_2 t} \end{aligned}} \quad \text{and} \quad \boxed{\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} -\left(\frac{c}{2m}\right) + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{Bmatrix}} \tag{10}$$

As before, the coefficients A and B are found from the initial position and velocity of the mass. In this case, we must solve two simultaneous equations for A and B.

$$\begin{aligned} x(0) &= \left(Ae^{\lambda_1 t} + Be^{\lambda_2 t}\right)_{t=0} &&\Rightarrow \boxed{A + B = x_0} \\ v(0) &= \left(A\lambda_1 e^{\lambda_1 t} + B\lambda_2 e^{\lambda_2 t}\right)_{t=0} &&\Rightarrow \boxed{\lambda_1 A + \lambda_2 B = v_0} \end{aligned} \tag{11}$$

Example:

Given: $m = 0.5$ (slug), $c = 15$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s)

Find: a) c_c the critical damping coefficient, and b) the displacement function $x(t)$.

Solution:

a) $c_c = 2m\sqrt{k/m} = 2\times 0.5\times\sqrt{50/0.5} = 10$ (lb-s/ft) so, the system is ***over-damped***.

b) $$\boxed{\begin{Bmatrix}\lambda_1\\ \lambda_2\end{Bmatrix} = \begin{Bmatrix} -\left(\frac{c}{2m}\right)+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}} \\ -\left(\frac{c}{2m}\right)-\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\end{Bmatrix} = \begin{Bmatrix} -\left(\frac{15}{2\times 0.5}\right)+\sqrt{\left(\frac{15}{2\times 0.5}\right)^2-\frac{50}{0.5}} \\ -\left(\frac{15}{2\times 0.5}\right)-\sqrt{\left(\frac{15}{2\times 0.5}\right)^2-\frac{50}{0.5}}\end{Bmatrix} \approx \begin{Bmatrix}-3.82\\ -26.18\end{Bmatrix}}$$

Simultaneous equations for the coefficients:

$$\begin{aligned} A + B &= 0.25 \\ -3.82A - 26.18B &= 5 \end{aligned} \quad\Rightarrow\quad \boxed{\begin{bmatrix} 1 & 1 \\ -3.82 & -26.18\end{bmatrix}\begin{Bmatrix}A\\ B\end{Bmatrix} = \begin{Bmatrix}0.25\\ 5\end{Bmatrix}}$$

Solving using Cramer's Rule:

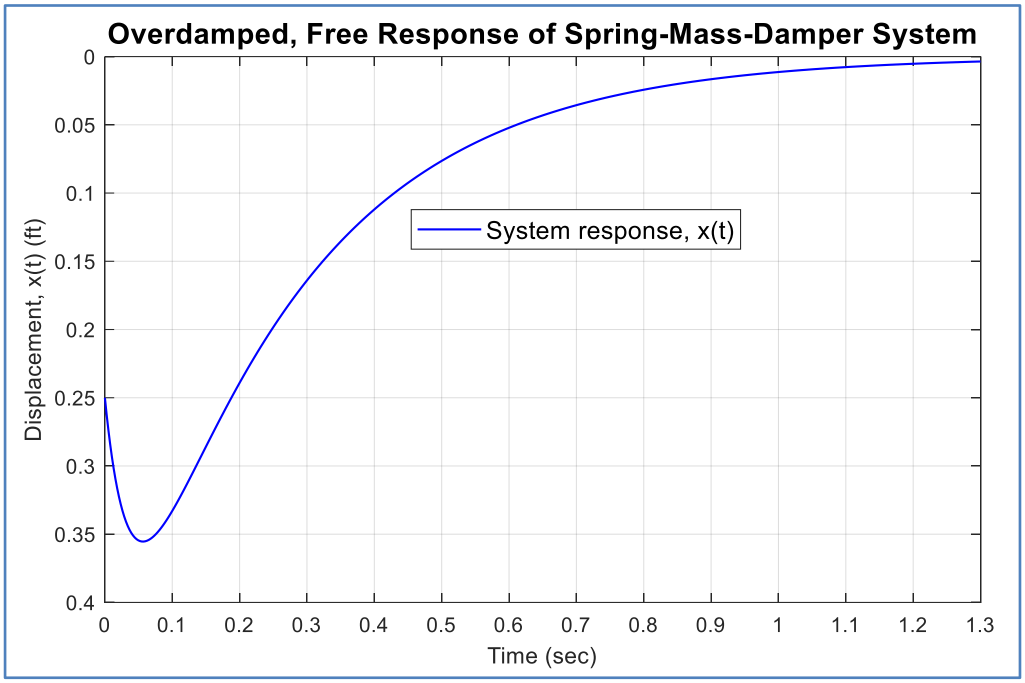
$$\boxed{A = \frac{\det\begin{bmatrix}0.25 & 1\\ 5 & -26.18\end{bmatrix}}{\det\begin{bmatrix}1 & 1\\ -3.82 & -26.18\end{bmatrix}} = \frac{-11.545}{-22.36} \approx 0.5163}$$

$$\boxed{B = \frac{\det\begin{bmatrix}1 & 0.25\\ -3.82 & 5\end{bmatrix}}{\det\begin{bmatrix}1 & 1\\ -3.82 & -26.18\end{bmatrix}} = \frac{5.955}{-22.36} \approx -0.2663}$$

So,

$$\boxed{x(t) \approx \left[(0.5163)e^{-3.82t} - (0.2663)e^{-26.18t}\right] \text{ (ft)}}$$ (See plot below)

Note from the plot that the mass ***moves away*** from the equilibrium for a ***short time*** due to its ***initial velocity***, but it then returns exponentially to the equilibrium position ($x = 0$).



Under-Damped Vibration: $\boxed{c < c_c}$

In this case, the mass ***oscillates*** as it returns to the equilibrium position. The amplitude of the oscillations reduce as time progresses. The position of the mass can be written as an exponential function times the sum of a sine and cosine function.

$$\boxed{x(t) = e^{-\left(c/2m\right)t}\left(A\sin(\omega_d t) + B\cos(\omega_d t)\right)} \quad \text{and} \quad \boxed{\begin{array}{l} A = \left[v_0 + \left(\frac{c}{2m}\right)x_0\right]\Big/\omega_d \\ B = x_0 \end{array}} \tag{12}$$

The frequency of the oscillation is $\boxed{\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}$. (13)

As previously noted, the sum of the sine and cosine terms can be written as a single phase-shifted sine function.

$$\boxed{x(t) = Me^{-\left(c/2m\right)t}\sin(\omega_d t + \phi)}$$

with $\boxed{M = \sqrt{A^2 + B^2}}$ and $\boxed{\phi = \tan^{-1}\left(B/A\right)}$.

Example:

Given: $m = 0.5$ (slug), $c = 5$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s)

Find: a) c_c the critical damping coefficient, and b) the displacement function $x(t)$.

Solution:

a) $c_c = 2m\sqrt{k/m} = 2 \times 0.5 \times \sqrt{50/0.5} = 10$ (lb-s/ft) so, the system is ***under-damped***.

b) $\boxed{\omega_d = \sqrt{\frac{50}{0.5} - \left(\frac{5}{2\times 0.5}\right)^2} = \sqrt{75} \approx 8.6603 \text{ (rad/s)}}$, $\boxed{B = x_0 = 0.25}$, and

$$\boxed{A = \left[v_0 + \left(\tfrac{c}{2m}\right)x_0\right]\Big/\omega_d \approx \left[5 + \left(\tfrac{5}{2\times 0.5}\right)0.25\right]\Big/8.6603 \approx 6.25/8.6603 \approx 0.7217},$$

$$\boxed{M \approx \sqrt{0.7217^2 + 0.25^2} \approx 0.7638}, \quad \boxed{\phi \approx \tan^{-1}\left(0.25/0.7217\right) \approx \begin{cases} 19.11 \text{ (deg)} \\ 0.3335 \text{ (rad)} \end{cases}}$$

So,

$\boxed{x(t) \approx 0.7638\, e^{-5t} \sin\left((8.6603)t + 0.3335\right) \text{ (ft)}}$ (See plot below)

Note from the plot that the mass initially moves away from the equilibrium position due to its initial velocity, then it oscillates about the equilibrium position as it returns to that position.