Elementary Engineering Mathematics
The Derivative of a Function as a Function

Previously, we learned about the meaning of the derivative of a function $f(x)$ at some arbitrary point x_0. The derivative $f'(x_0)$ is simply the slope of the tangent line at x_0. We now consider the function $f'(x)$ or $\frac{df}{dx}(x)$ which consists of the derivatives of $f(x)$ at all points within the range of x. The following table gives the derivatives of some common functions used in engineering.

Name	Function, $f(x)$	Derivative, $f'(x) = \frac{df}{dx}(x)$
Constant	a	0
Polynomial terms	$a x^n$	$n a x^{n-1}$
Exponential	e^{ax}	$a e^{ax}$
Sine	$\sin(ax)$	$a \cos(ax)$
Cosine	$\cos(ax)$	$-a \sin(ax)$

To evaluate the derivative at some point x_0, we can simply evaluate the derivative function $f'(x)$ at x_0.

These results can be extended to combinations of functions by using the following ***rules*** for differentiation.

	Name	Formula
1	Summation rule	$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$
2	Multiplication by a constant, a	$\frac{d}{dx}\big(a\, f(x)\big) = a\ f'(x)$
3	Product rule	$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x)+f(x)g'(x)$
4	Chain rule	$\frac{d}{dx}\big(f(y(x))\big) = \frac{df}{dy}\frac{dy}{dx} = \frac{df}{dy}\ y'(x)$

Example 1:

Given: The path of a golf ball as a function of horizontal position

$$y = f(x) = 73.54\left(\frac{x}{61.71}\right) - 16.1\left(\frac{x}{61.71}\right)^2 = 1.1917\,x - \left(4.2278\times10^{-3}\right)x^2 \qquad (1)$$

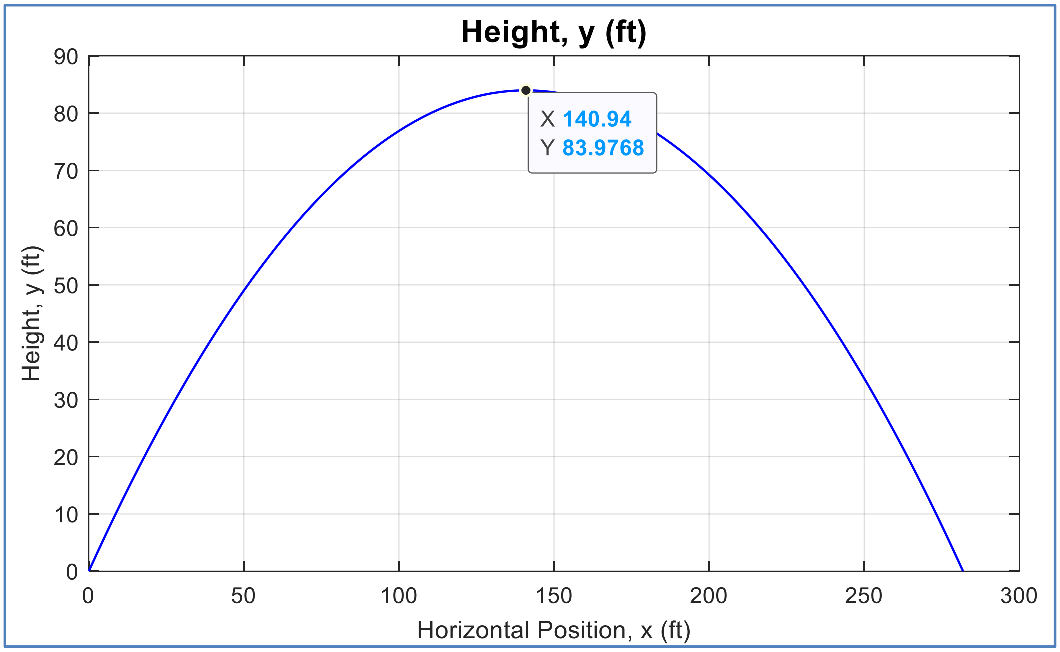


Figure 1. Height of Golf Ball as a Function of Distance, x

Recall that the ***velocity*** of the ball is in the direction of the tangent line.

Find: (a) The ***derivative function*** $f'(x)$, (b) the derivative at $x = x_0 = 50$ (ft) and $x = x_0 = 200$ (ft), (c) the maximum height of the ball, (d) a plot of function $f'(x)$, and (e) the second derivative function $f''(x) = df'/dx$.

Solution:

(a) Using rule 1 above, we can find the derivative function $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(1.1917\,x - \left(4.2278\times10^{-3}\right)x^2\right) = \frac{d}{dx}(1.1917\,x) - \frac{d}{dx}\left(\left(4.2278\times10^{-3}\right)x^2\right)$$

$$= 1.1917\frac{d}{dx}(x) - \left(4.2278\times10^{-3}\right)\frac{d}{dx}\left(x^2\right)$$

$$= (1.1917\times1) - \left(4.2278\times10^{-3}\right)(2x)$$

or

$$f'(x) = 1.1917 - \left(8.4556\times10^{-3}\right)x \qquad (2)$$

(b) We can use the derivative function in Eq. (2) to find the derivative of the quadratic function at any point in the domain of x.

At $x = x_0 = 50$ (ft): $\boxed{f'(x)\big|_{x=50} = 1.1917 - \left(8.4556\times10^{-3}\right)(50) \approx 0.76892 \approx 0.769}$

At $x = x_0 = 200$ (ft): $\boxed{f'(x)\big|_{x=200} = 1.1917 - \left(8.4556\times10^{-3}\right)(200) \approx -0.49942 \approx -0.499}$

(c) The maximum height of the ball occurs where the slope of the tangent line is zero.

$$f'(\hat{x}) = 0 = 1.1917 - \left(8.4556\times10^{-3}\right)\hat{x} \quad \Rightarrow \quad \boxed{\hat{x} = 1.1917/8.4556e-3 = 140.94 \text{ (ft)}}$$

$$y_{\max} = f(\hat{x}) = 1.1917\,\hat{x} - \left(4.2278\times10^{-3}\right)\hat{x}^2 \quad \Rightarrow \quad \boxed{y_{\max} = 83.977 \approx 84 \text{ (ft)}}$$

(d) The plot of $f'(x)$ indicates that the slope of $f(x)$ is positive over the first half of the range of x, negative over the second half, and zero at the maximum height of the ball. This confirms the fact that the function reached a ***maximum*** at this point, and ***not*** a ***minimum***. When $f'(x)$ is positive, $f(x)$ is increasing, and when $f'(x)$ is negative, $f(x)$ is decreasing.

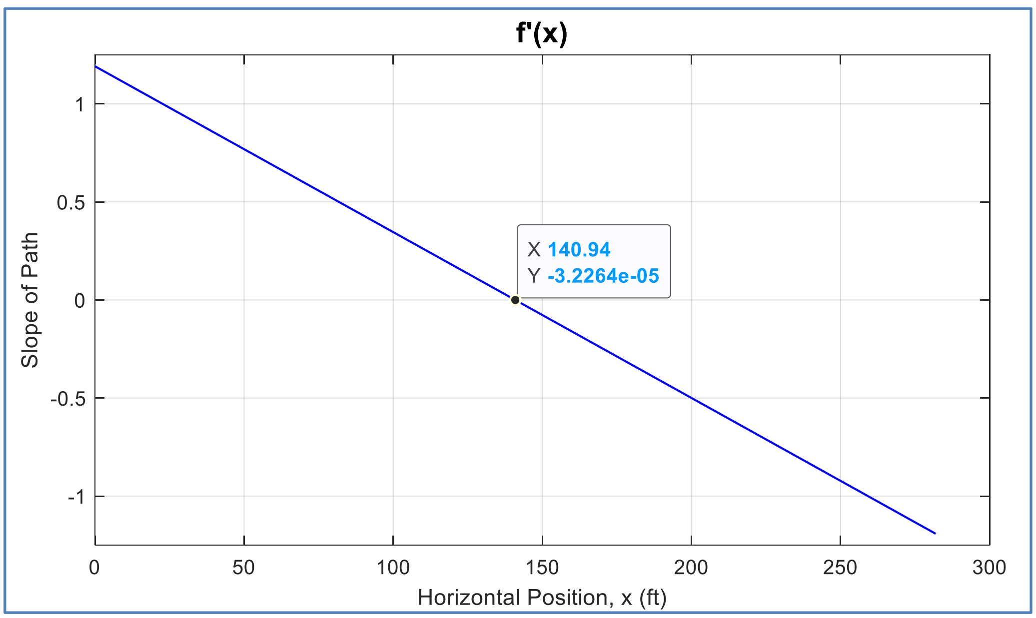


Figure 2. Slope of the Path of the Golf Ball as a Function of x

(e) Using rule 1, $\boxed{f''(x) = \dfrac{d}{dx}\left[1.1917 - \left(8.4556\times10^{-3}\right)x\right] = -8.4556\times10^{-3} < 0}$. This again confirms that the function is maximum at this point. If $f''(\hat{x}) < 0$, the function is at a ***local maximum***, and if $f''(\hat{x}) > 0$, the function is at a ***local minimum***.

Example 2:

Given: The height of a golf ball as a function of time $y = f(t) = 73.54t - 16.1t^2$

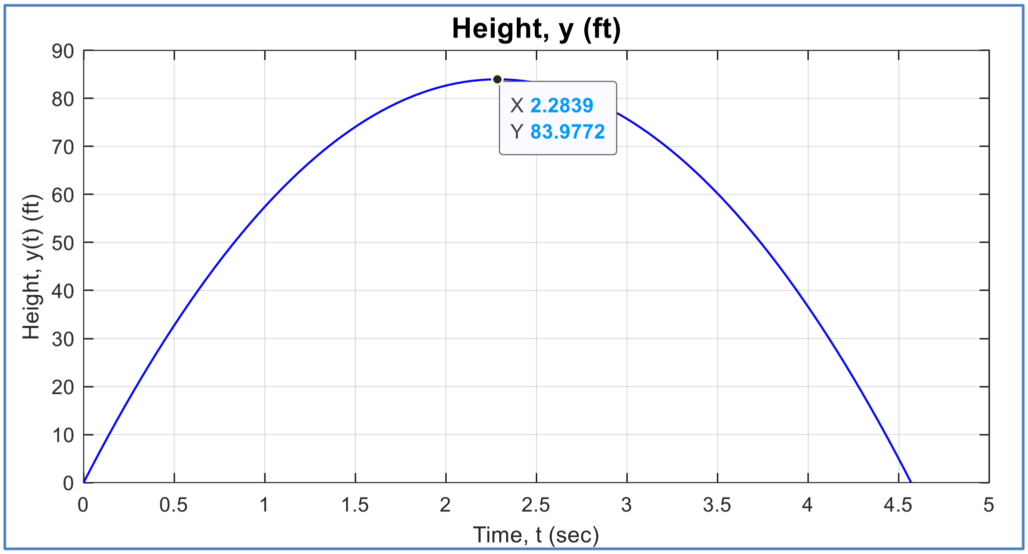


Figure 3. Height of the Ball as a Function of Time

Find: (a) the derivative function $f'(t)$, (b) the slope of $f(t)$ at $t = 0.5$ (sec), and (c) the time $\hat{t}$ when the ball reaches maximum height.

Solution:

(a) We can again use rule 1 to find $f'(t)$.

$$\frac{dy}{dt} = f'(t) = (73.54 \times 1) - 16.1(2t) = 73.54 - 32.2t$$

(b) When $t = 0.5$ (sec), $\left.\frac{dy}{dt}\right|_{t=0.5} = \left. f'(t)\right|_{t=0.5} = 73.54 - (32.2 \times 0.5) \approx 57.44 \text{ (ft/sec)}$. This is the velocity of the ball in the Y-direction at this instant.

(c) To find time $\hat{t}$, we set $f'(\hat{t}) = 0$, and solve

$$f'(\hat{t}) = 0 = 73.54 - 32.2\hat{t} \quad \Rightarrow \quad \hat{t} = 73.54/32.2 \approx 2.2839 \approx 2.28 \text{ (sec)}$$

Example 3:

Given: The horizontal position and height of a golf ball are functions of time

$$\boxed{x(t) = 61.71t} \qquad \boxed{y(t) = 73.54t - 16.1t^2} \qquad (3)$$

Find: The velocity vector of the ball when $x = x_0 = 50$ (ft).

Solution:

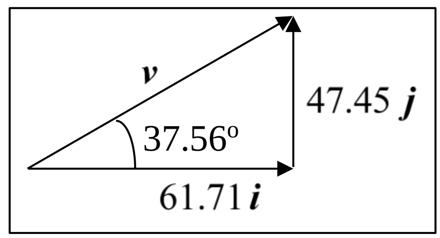
The components of the velocity of the ball in the X and Y directions are given by the derivatives of Eqs. (3) with respect to time. We must first find the time $\hat{t}$ required to get to $x = x_0 = 50$ (ft).

$$x(\hat{t}) = 50 = 61.71\hat{t} \quad \Rightarrow \quad \boxed{\hat{t} = 50/61.71 \approx 0.81024 \text{ (sec)}}$$

$$\boxed{v_x(t) = \frac{dx}{dt} = \dot{x}(t) = 61.71 \times (1) \approx 61.71 \text{ (ft/sec)}} \quad \text{(same at all } t\text{)}$$

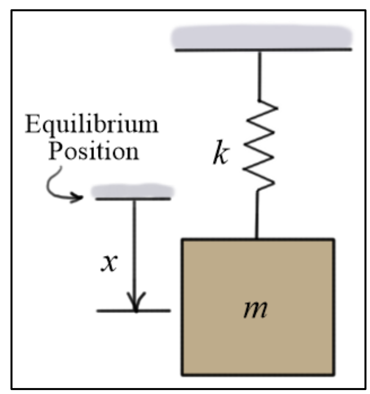
$$\boxed{v_y(t) = \frac{dy}{dt} = \dot{y}(t) = 73.54 - 32.2t}$$

$$\Rightarrow \boxed{v_y(\hat{t}) = 73.54 - (32.2 \times 0.81024) \approx 47.45 \text{ (ft/sec)}}$$



So, at $x = x_0 = 50$ (ft), we have $\boxed{\underset{\sim}{v} = 61.71\underset{\sim}{i} + 47.45\underset{\sim}{j} \text{ (ft/sec)}}$

Example 4: Undamped, free vibration



Given: In response to the initial position x_0 and initial velocity v_0, the undamped spring-mass-damper system has displacement function

$$\boxed{x(t) = \tfrac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)} \qquad \boxed{\omega = \sqrt{\tfrac{k}{m}}}$$

Find: (a) the velocity function $v(t) = \dot{x}(t)$, (b) the acceleration function $a(t) = \dot{v}(t) = \ddot{x}(t)$, (c) $x(0)$, $v(0)$, and $a(0)$, the position, velocity and acceleration of the mass at $t = 0$, and (d) the times when $x(t)$ has a maximum or minimum if $v_0 = 0$, and verify which are maxima and which are minima.

Solution:

(a) Using rules 1 and 2:

$$v(t) = \dot{x}(t) = \frac{d}{dt}\left[\tfrac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)\right] = \frac{d}{dt}\left[\tfrac{v_0}{\omega}\sin(\omega t)\right] + \frac{d}{dt}\left[x_0\cos(\omega t)\right]$$

$$= \tfrac{v_0}{\not\omega}\not\omega\cos(\omega t) - x_0\omega\sin(\omega t)$$

$$\boxed{v(t) = v_0\cos(\omega t) - x_0\omega\sin(\omega t)}$$

(b) Again, using rules 1 and 2:

$$a(t) = \ddot{x}(t) = \dot{v}(t) = \frac{d}{dt}\left[v_0\cos(\omega t) - x_0\omega\sin(\omega t)\right]$$

$$= \frac{d}{dt}\left[v_0\cos(\omega t)\right] - \frac{d}{dt}\left[x_0\omega\sin(\omega t)\right]$$

$$\boxed{a(t) = -v_0\,\omega\,\sin(\omega t) - x_0\omega^2\cos(\omega t)}$$

(c) $\boxed{x(0) = \tfrac{v_0}{\omega}\sin(0) + x_0\cos(0) = x_0}$ $\boxed{v(0) = v_0\cos(0) - x_0\omega\sin(0) = v_0}$ (checks)

$$\boxed{a(0) = -v_0\,\omega\,\sin(0) - x_0\omega^2\cos(0) = -x_0\omega^2}$$

(d) If $v_0 = 0$, the velocity and acceleration of the mass are $\boxed{v(t) = -x_0\,\omega\,\sin(\omega t)}$ and $\boxed{a(t) = -x_0\omega^2\cos(\omega t)}$. The position has a maximum or minimum at times $\hat{t}$ when the velocity $v(\hat{t}) = 0$. So, the position will be a maximum or minimum when $\omega\hat{t} = n\pi$ or $\boxed{\hat{t} = n\pi/\omega}$ $(n = 0,1,2,...)$. The results are summarized in the following table.

n	$v(\hat{t})$	$a(\hat{t})$	Type
$0,2,4,\ldots$	zero	negative	maximum
$1,3,5,\ldots$	zero	positive	minimum

The figure below shows the position, velocity, and acceleration functions for $x_0 = 0.25$ (ft) and $\omega = 2\pi/3$ (rad/s). When the velocity is zero, the displacement is either a maximum or minimum. It is a maximum when the acceleration is negative, and it is a minimum when the acceleration is positive (as indicated in the table above).

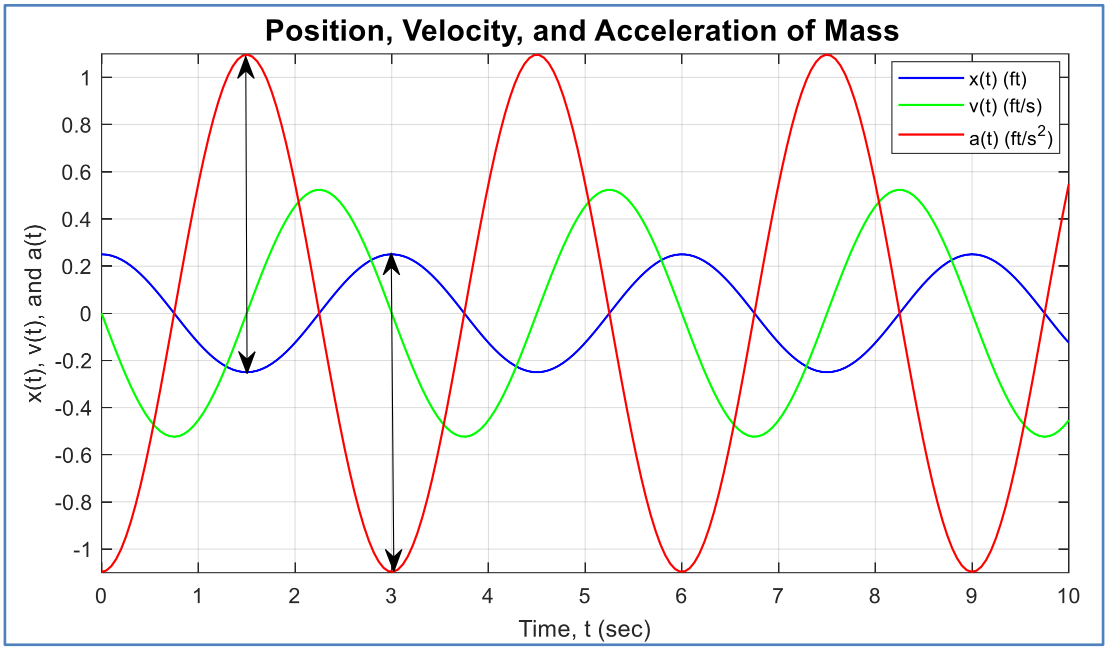
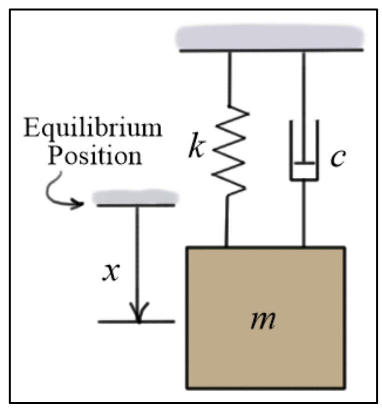


Figure 4. Position, Velocity, and Acceleration of Mass Due to Initial Displacement



<u>Example 5</u>: Over-damped, free vibration

<u>Given</u>: In response to the initial conditions $x_0 = 0.25$ (ft) and $v_0 = 5$ (ft/s), the over-damped spring-mass-damper system has displacement

$$\boxed{x(t) = (0.5163)e^{-3.82t} - (0.2663)e^{-26.18t} \text{ (ft)}}$$

<u>Find</u>: (a) the velocity function $v(t) = \dot{x}(t)$; (b) the acceleration function $a(t) = \dot{v}(t) = \ddot{x}(t)$; (c) $x(0)$, $v(0)$, and $a(0)$, the position, velocity and acceleration of the mass at $t = 0$; and (d) find the time when the displacement is maximum.

<u>Solution</u>:

(a) Using rules 1 and 2, we find $v(t)$

$$v(t) = \dot{x}(t) = \frac{d}{dt}\left((0.5163)e^{-3.82t} - (0.2663)e^{-26.18t}\right)$$

$$= \frac{d}{dt}\left[(0.5163)e^{-3.82t}\right] - \frac{d}{dt}\left[(0.2663)e^{-26.18t}\right]$$

$$= (0.5163)(-3.82)e^{-3.82t} - (0.2663)(-26.18)e^{-26.18t}$$

$$\boxed{v(t) = -1.9723e^{-3.82t} + 6.9717e^{-26.18t} \text{ (ft/s)}}$$

(b) Using rules 1 and 2 again, we find $a(t)$

$$a(t) = \ddot{x}(t) = \dot{v}(t) = \frac{d}{dt}\left[-1.9723e^{-3.82t} + 6.9717e^{-26.18t}\right]$$

$$= (-1.9723)(-3.82)e^{-3.82t} + (6.9717)(-26.18)e^{-26.18t}$$

$$\boxed{a(t) = 7.5342e^{-3.82t} - 182.52e^{-26.18t} \text{ (ft/s}^2\text{)}}$$

(c) $\boxed{x(0) = \left((0.5163)e^{0} - (0.2663)e^{0}\right) = 0.5163 - 0.2663 = 0.25 \text{ (ft)}}$ (checks)

$\boxed{v(0) = -1.9723e^{0} + 6.9717e^{0} = 6.9717 - 1.9723 = 4.9994 \approx 5 \text{ (ft/s)}}$ (checks)

$\boxed{a(0) = 7.5342e^{0} - 182.52e^{0} = 7.5342 - 182.52 \approx -175 \text{ (ft/s}^2\text{)}}$

(d) To find the time when the displacement is maximum, we set $dx(t)/dt = v(t) = 0$.

$$v(t) = -1.9723e^{-3.82t} + 6.9717e^{-26.18t} = 0 \quad \Rightarrow 6.9717e^{-26.18t} = 1.9723e^{-3.82t}$$

$$\Rightarrow \frac{6.9717}{1.9723} = 3.5348 = \frac{e^{-3.82t}}{e^{-26.18t}} = e^{22.36t} \quad \Rightarrow \ln\left(3.5348\right) = \ln\left(e^{22.36t}\right) = 22.36\,t$$

$$\boxed{t = \ln\left(3.5348\right)/22.36 = 0.0565 \text{ (sec)}}$$

$$\boxed{x(0.0565) = (0.5163)e^{-0.21583} - (0.2663)e^{-1.4792} = 0.355 \text{ (ft)}}$$

$\boxed{a(0.0565) = 7.5342e^{-0.21583} - 182.52e^{-1.4792} = -35.5 \text{ (ft/s}^2\text{)}}$ (indicates maximum)

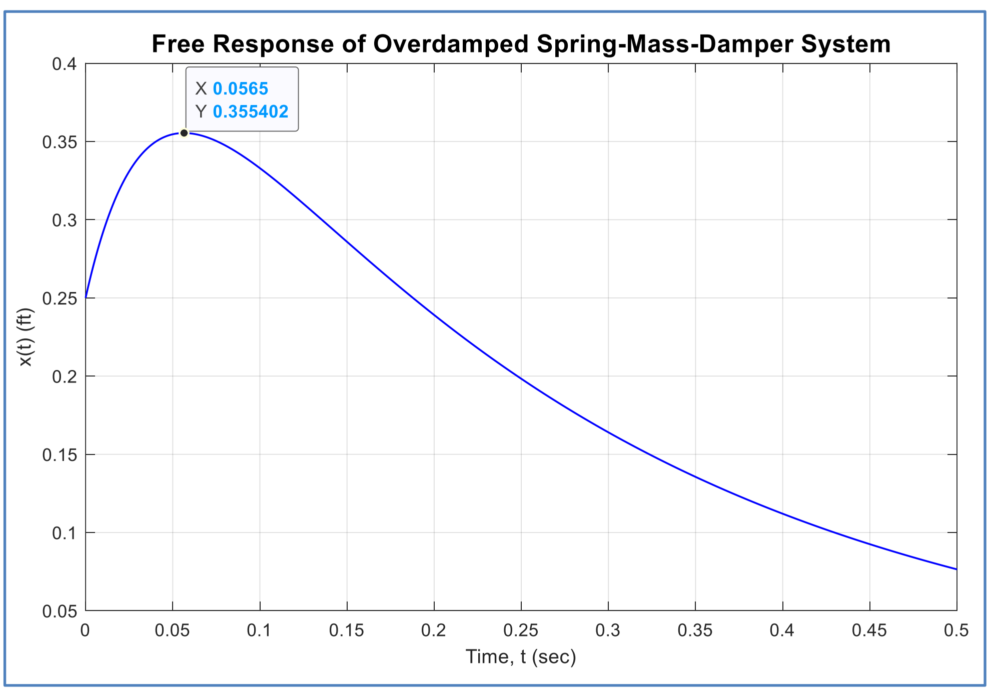
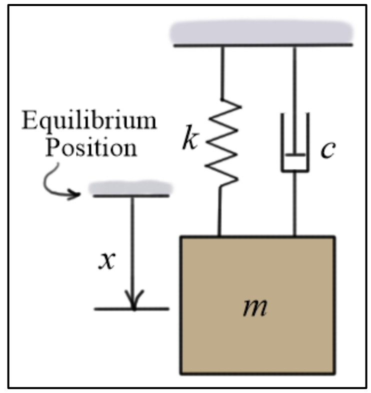


Figure 5. Response of Overdamped Spring-Mass-Damper System

Example 6: Under-damped, free vibration



Given: In response to the initial conditions $x_0 = 0.25$ (ft) and $v_0 = 5$ (ft/s), the under-damped spring-mass-damper system has displacement

$$x(t) = e^{-5t}\left[0.7217\sin\left(\sqrt{75}\,t\right) + 0.25\cos\left(\sqrt{75}\,t\right)\right] \text{ (ft)}$$

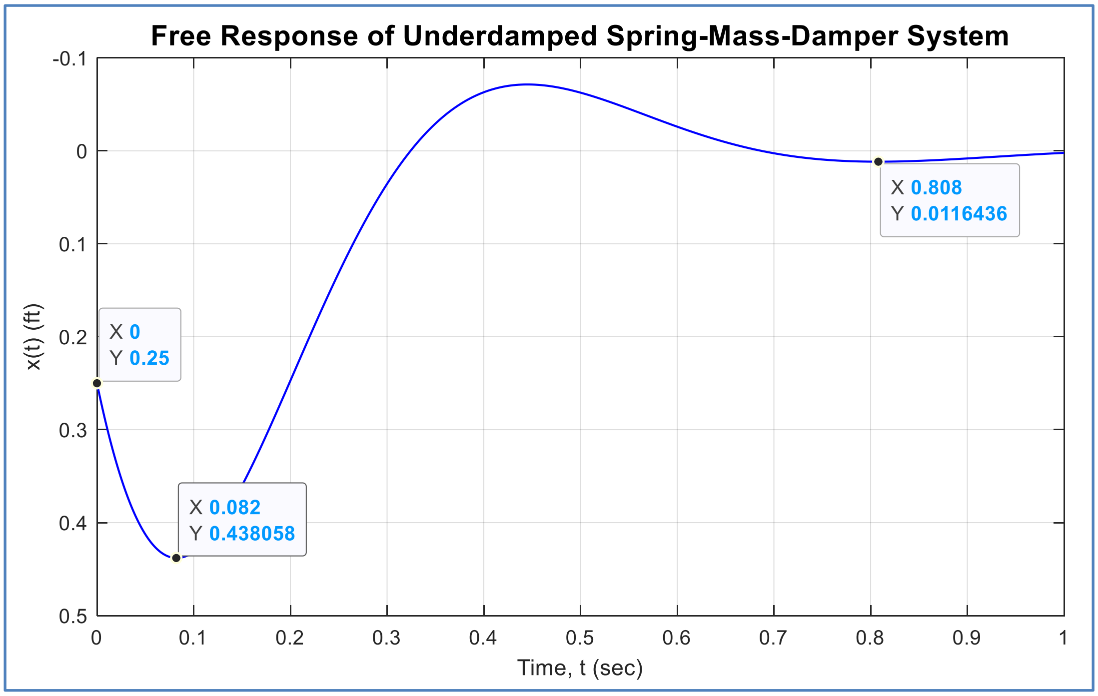


Figure 6. Response of Underdamped Spring-Mass-Damper System

Find: (a) the velocity function $v(t) = \dot{x}(t)$, (b) the acceleration function $a(t) = \dot{v}(t) = \ddot{x}(t)$, and (c) $x(0)$, $v(0)$, and $a(0)$, the position, velocity and acceleration of the mass at $t = 0$.

Solution:

(a) Using rules 2, 3 and 4, we find the velocity function $v(t)$

$$\begin{aligned} v(t) &= \dot{x}(t) \\ &= \frac{d}{dt}\left(e^{-5t}\left[0.7217\sin\left(\sqrt{75}\,t\right) + 0.25\cos\left(\sqrt{75}\,t\right)\right]\right) \\ &= \frac{d}{dt}\left(e^{-5t}\right) \times \left[0.7217\sin\left(\sqrt{75}\,t\right) + 0.25\cos\left(\sqrt{75}\,t\right)\right] \\ &\quad + e^{-5t}\left(\frac{d}{dt}\left[0.7217\sin\left(\sqrt{75}\,t\right) + 0.25\cos\left(\sqrt{75}\,t\right)\right]\right) \\ &= \cdots \end{aligned}$$

$$= -5e^{-5t}\left[0.7217\,\sin\left(\sqrt{75}\,t\right) + 0.25\,\cos\left(\sqrt{75}\,t\right)\right]$$
$$+\, e^{-5t}\left((0.7217)\sqrt{75}\,\cos\left(\sqrt{75}\,t\right) - (0.25)\sqrt{75}\sin\left(\sqrt{75}\,t\right)\right)$$
$$= e^{-5t}\left[-\left((5\times 0.7217) + (0.25\times\sqrt{75})\right)\sin\left(\sqrt{75}\,t\right)\right]$$
$$+\, e^{-5t}\left[\left(\left(0.7217\sqrt{75}\right) - (5\times 0.25)\right)\cos\left(\sqrt{75}\,t\right)\right]$$
$$\Rightarrow \boxed{v(t) = e^{-5t}\left[-5.7736\sin\left(\sqrt{75}\,t\right) + 5\cos\left(\sqrt{75}\,t\right)\right]\ \text{(ft/s)}}$$

(b) Using rules 2, 3 and 4, we find the acceleration function $a(t)$

$$a(t) = \ddot{x}(t) = \dot{v}(t)$$
$$= \frac{d}{dt}\left(e^{-5t}\left[-5.7736\sin\left(\sqrt{75}\,t\right) + 5\cos\left(\sqrt{75}\,t\right)\right]\right)$$
$$= \frac{d}{dt}\left(e^{-5t}\right)\times\left[-5.7736\,\sin\left(\sqrt{75}\,t\right) + 5\,\cos\left(\sqrt{75}\,t\right)\right]$$
$$+\, e^{-5t}\left(\frac{d}{dt}\left[-5.7736\,\sin\left(\sqrt{75}\,t\right) + 5\,\cos\left(\sqrt{75}\,t\right)\right]\right)$$
$$= -5e^{-5t}\left[-5.7736\,\sin\left(\sqrt{75}\,t\right) + 5\,\cos\left(\sqrt{75}\,t\right)\right]$$
$$+\, e^{-5t}\left(\left(-5.7736\sqrt{75}\right)\cos\left(\sqrt{75}\,t\right) - (5)\sqrt{75}\sin\left(\sqrt{75}\,t\right)\right)$$
$$= e^{-5t}\left[\left((5\times 5.7736) - (5\sqrt{75})\right)\sin\left(\sqrt{75}\,t\right)\right]$$
$$+\, e^{-5t}\left[-\left(\left(5.7736\sqrt{75}\right) + (5\times 5)\right)\cos\left(\sqrt{75}\,t\right)\right]$$
$$\Rightarrow \boxed{a(t) = -e^{-5t}\left[14.43\sin\left(\sqrt{75}\,t\right) + 75\cos\left(\sqrt{75}\,t\right)\right]\ \text{(ft/s}^2\text{)}}$$

(c) $\boxed{x(0) = e^{0}\left[0.7217\,\sin(0) + 0.25\,\cos(0)\right] = 0.25\ \text{(ft)}}$ (checks)

$\boxed{v(0) = e^{0}\left[-5.7736\sin(0) + 5\cos(0)\right] = 5\ \text{(ft/s)}}$ (checks)

$\boxed{a(0) = -e^{0}\left[14.43\sin(0) + 75\cos(0)\right] = -75\ \text{(ft/s}^2\text{)}}$

At time $t = 0$, the mass is ***moving downward*** at $5\ \text{(ft/s)}$, but it is ***slowing down*** at a rate of $75\ \text{(ft/s}^2\text{)}$. According to Figure 6, the mass reaches a ***zero downward velocity*** at about $t \approx 0.082\ \text{(sec)}$.