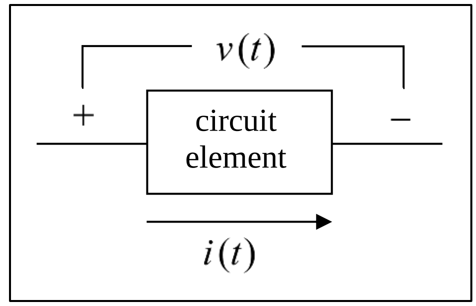
**Elementary Engineering Mathematics**
**Application of Derivatives in Electrical Engineering**

The diagram shows a typical element (resistor, capacitor, inductor, etc.) in an electrical circuit. Here, $v(t)$ represents the voltage across the element, and $i(t)$ represents the current flowing through the element. Both generally are functions of time, $t$. For any such element, the following equations apply.




$$\boxed{v(t)=\frac{dw}{dq}} \qquad \begin{cases} v(t) \text{ is the voltage (volts)} \\ w(t) \text{ is the energy (joules)} \\ q(t) \text{ is the charge (coulombs)} \end{cases}$$

$$\boxed{i(t)=\frac{dq}{dt}} \qquad \begin{cases} i(t) \text{ is the current (amps)} \\ t \text{ is the time (sec)} \end{cases}$$

$$\boxed{p(t)=\frac{dw}{dt}=\left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right)=v(t)\cdot i(t)} \qquad \{p(t) \text{ is the power (joules/sec) or (watts)}$$

This last equation is an application of the ***chain rule***.

Example 1:

Given: The ***charge*** and ***voltage*** for a given circuit element are given by the following equations:

$$\boxed{q(t)=\tfrac{1}{50}\sin\left(250\pi t\right) \text{ (coulombs)}} \text{ and } \boxed{v(t)=100\sin\left(250\,\pi\,t\right) \text{ (volts)}}$$

Find: a) $i(t)$ the ***current*** passing through the element, b) $p(t)$ the ***power*** dissipated by the element, and c) $p_{\max}$ the ***maximum power*** dissipated by the element.

Solution:

a) $i(t)=\dfrac{dq}{dt}=\dfrac{d}{dt}\left(\tfrac{1}{50}\sin\left(250\pi t\right)\right)=\tfrac{1}{50}\times\cos\left(250\pi t\right)\times 250\pi=\boxed{5\pi\cos\left(250\pi t\right) \text{ (amps)}}$

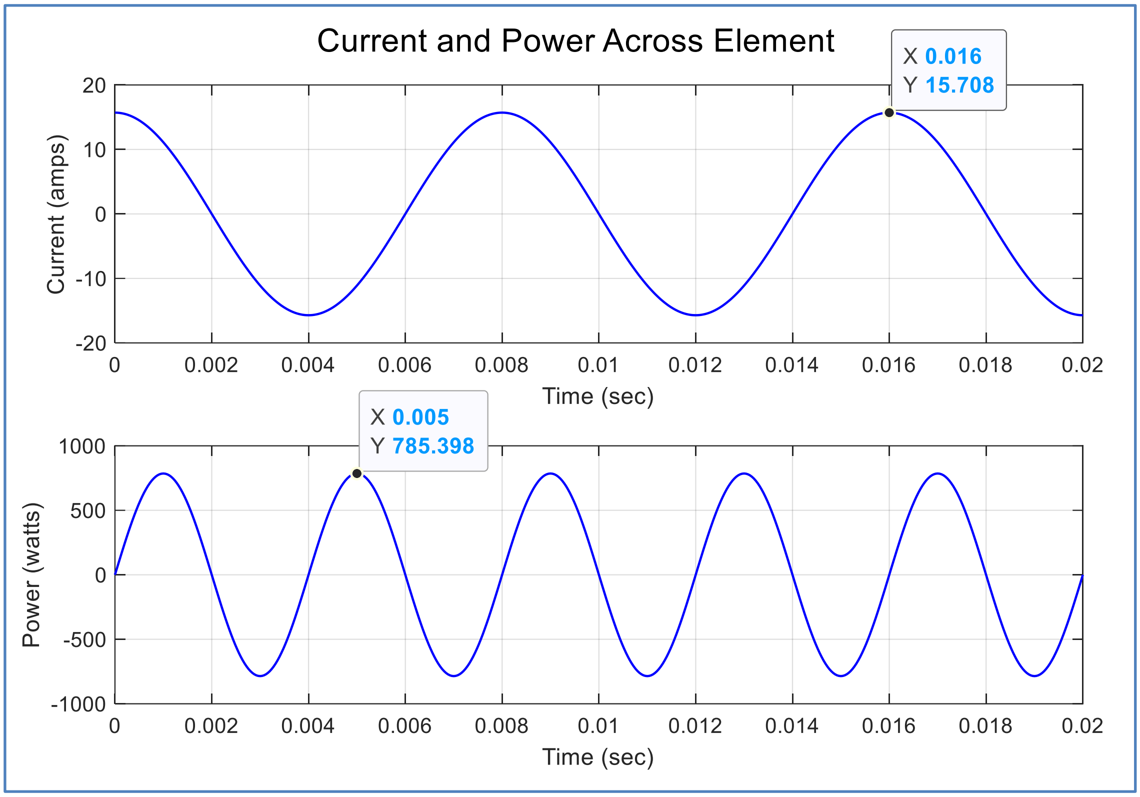
b) $p(t)=v(t)\cdot i(t)=\left(100\sin\left(250\pi t\right)\right)\left(5\pi\cos\left(250\pi t\right)\right)$

$$=\boxed{500\pi\sin\left(250\pi t\right)\cos\left(250\pi t\right) \text{ (watts)}}$$

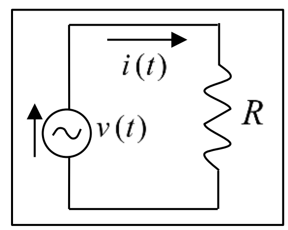
c) To find maximum power, we use the ***trigonometric identity***: $\boxed{2\sin\left(\theta\right)\cos\left(\theta\right)=\sin\left(2\theta\right)}$

$$p(t)=500\pi\sin\left(250\pi t\right)\cos\left(250\pi t\right)=\tfrac{1}{2}\left(500\pi\sin\left(500\pi t\right)\right)=\boxed{250\pi\sin\left(500\pi t\right) \text{ (watts)}}$$

$$\Rightarrow \boxed{p_{\max}=250\pi\approx 785 \text{ (watts)}}$$

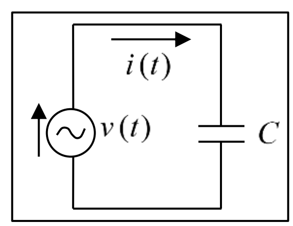



## Current–Voltage Relationships for Resistors, Capacitors, and Inductors




The voltage across and the current through a ***resistor*** are related simply by its resistance.

$$\boxed{v(t) = R\,i(t)} \quad \text{or} \quad \boxed{i(t) = v(t)/R}$$

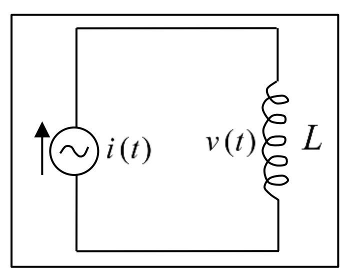



Given a voltage $v(t)$ applied to a ***capacitor***, the corresponding current $i(t)$ can be calculated as

$$\boxed{i(t) = C\frac{dv}{dt}}$$

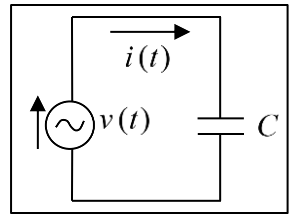
Given the current $i(t)$ passing through an ***inductor***, the corresponding voltage $v(t)$ can be calculated as

$$\boxed{v(t) = L\frac{di}{dt}}$$



Example 2:

Given: A voltage $v(t)=110\cos(120\pi t)$ (volts) is applied to a capacitor with $C=100$ ($\mu$f ).




Find: a) $i(t)$ the current through the capacitor, and b) $p_{\max}$ the maximum power.

Solution:

a) The current can be found by differentiating the voltage.

$$i(t)=C\frac{dv}{dt}=\left(100\times10^{-6}\right)\frac{d}{dt}\left(110\cos(120\pi t)\right)$$

$$=\left(100\times10^{-6}\right)(110)(120\pi)\left(-\sin(120\pi t)\right)$$

$$=\boxed{-4.147\sin(120\pi t)\ \text{(amps)}}$$

b) $p(t)=v(t)\cdot i(t)=\left(110\cos(120\pi t)\right)\left(-4.147\sin(120\pi t)\right)$

$$=-456\sin(120\pi t)\cos(120\pi t)$$

$$=-456\times\tfrac{1}{2}\sin(240\pi t)$$

$$=\boxed{-228\sin(240\pi t)\ \text{(watts)}}\quad\Rightarrow\quad\boxed{p_{\max}=228\ \text{(watts)}}$$

Alternate Solution for part (a): (without using derivatives)

We could have solved this problem using ***complex numbers***.

$$Z_C=-j/\omega C=-j\big/(120\pi)\left(100\times10^{-6}\right)=-j\left(10^6\right)\big/(120\pi)(100)$$

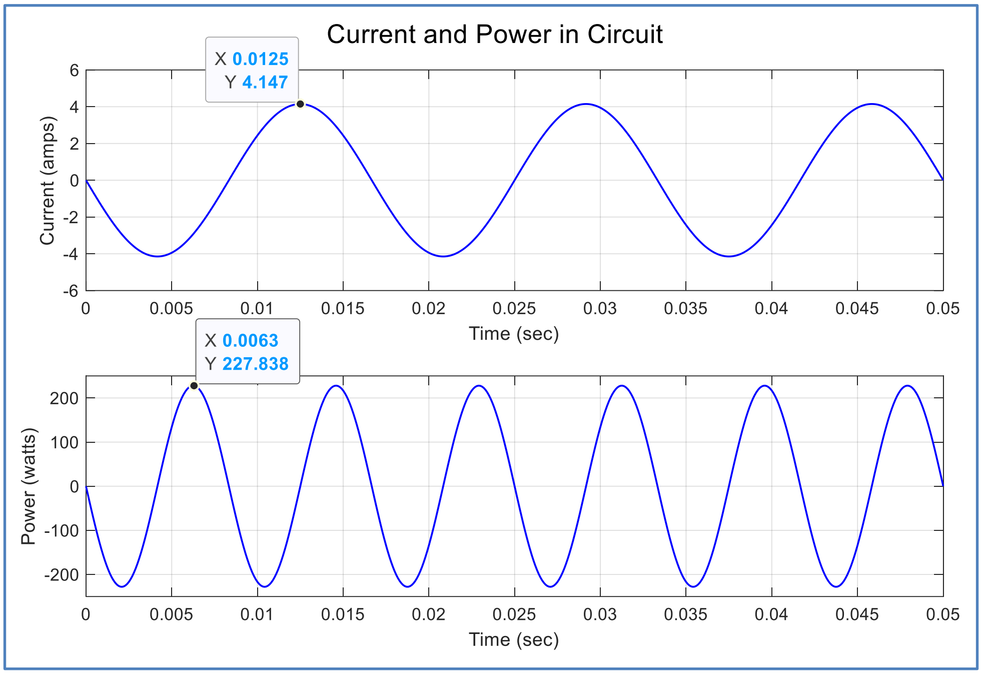
$$\approx -j\,26.526\ \text{(ohms)}\approx 26.526\angle\left(-90^{\circ}\right)$$

$$\boxed{I\approx\frac{110\angle\left(0^{\circ}\right)}{26.526\angle\left(-90^{\circ}\right)}\approx 4.147\angle\left(90^{\circ}\right)}$$

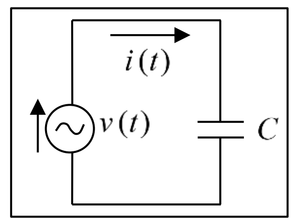
$$\Rightarrow i(t)\approx 4.147\cos\left(120\pi t+90^{\circ}\right)\ \text{(amps)}$$

$$\approx 4.147\left(\cos(120\pi t)\underbrace{\cos\left(90^{\circ}\right)}_{0}-\sin(120\pi t)\underbrace{\sin\left(90^{\circ}\right)}_{1}\right)$$

$$\Rightarrow\boxed{i(t)\approx -4.147\sin(120\pi t)\ \text{(amps)}}$$




Example 3:




Given: A voltage $v(t) = 110e^{-10t}\cos(120\pi t)$ (volts) is applied to a capacitor with $C = 100$ ($\mu$f).

Find: $i(t)$, the current through the capacitor

Solution:

To find the current we differentiate the voltage using the product and chain rules.

$$i(t) = C\frac{dv}{dt} = \left(100\times10^{-6}\right)\frac{d}{dt}\left(110e^{-10t}\cos(120\pi t)\right)$$

$$= \left(100\times10^{-6}\right)(110)\left[\left(\frac{d}{dt}\left(e^{-10t}\right)\times\left(\cos(120\pi t)\right)\right)+\left(\left(e^{-10t}\right)\times\frac{d}{dt}\left(\cos(120\pi t)\right)\right)\right]$$

$$= 0.011\left[\left(-10e^{-10t}\cos(120\pi t)\right)+\left(-120\pi e^{-10t}\sin(120\pi t)\right)\right]$$

$$\Rightarrow \boxed{i(t) \approx -0.011e^{-10t}\left[10\cos(120\pi t)+120\pi\sin(120\pi t)\right] \text{ (amps)}}$$

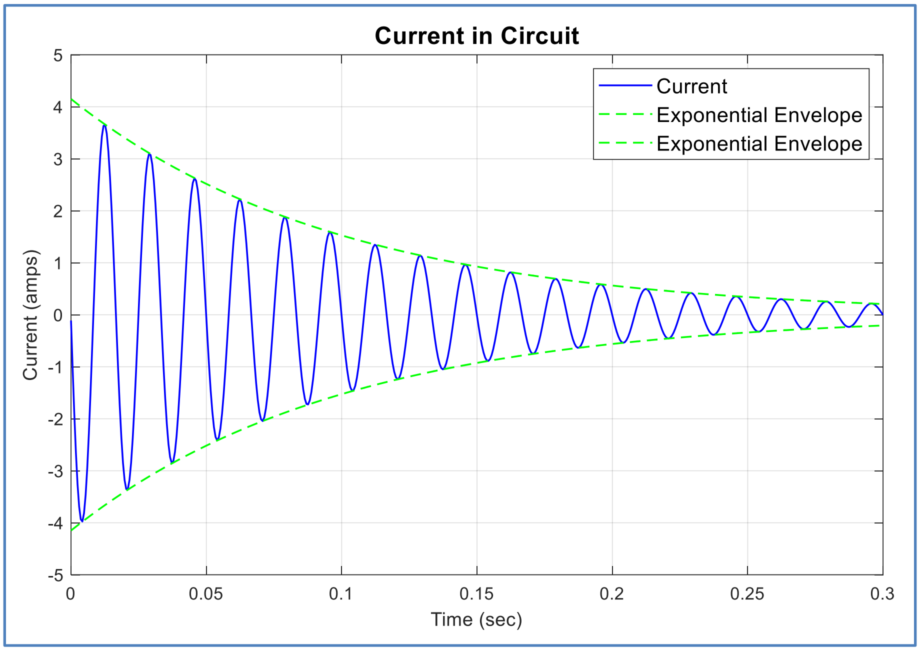
The term in square brackets can be reduced to a ***single***, ***phase-shifted sine*** or ***cosine*** function. For example,

$$\boxed{120\pi\sin(120\pi t)+10\cos(120\pi t) = M\cos(120\pi t+\phi)}$$

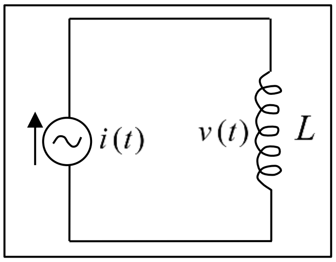
where

$\boxed{M = \sqrt{(120\pi)^2+10^2} \approx 377.1}$ and $\boxed{\phi = \tan^{-1}(-120\pi/10) = -88.48^{\circ} = -1.544 \text{ (rad)}}$.

$$\Rightarrow \boxed{i(t) \approx -4.15e^{-10t}\cos(120\pi t-1.544) \text{ (amps)}}$$




Example 4:




Given: Current $i(t) = 5te^{-10t}$ (amps) is applied to an inductor with $L = 250$ (mh).

Find: $v(t)$, the voltage across the inductor.

Solution:

To find the voltage, we differentiate the current using the product and chain rules.

$$v(t) = L\frac{di}{dt} = 0.25\frac{d}{dt}\left(5te^{-10t}\right) = \tfrac{1}{4}\left(\frac{d}{dt}(5t)\right)\left(e^{-10t}\right) + \tfrac{1}{4}\left((5t)\frac{d}{dt}\left(e^{-10t}\right)\right)$$

$$= \tfrac{1}{4}\left(5e^{-10t}\right) + \tfrac{1}{4}\left(5t\left(-10e^{-10t}\right)\right) = \tfrac{1}{4}\left(5e^{-10t} - 50te^{-10t}\right)$$

$$\Rightarrow \boxed{v(t) = \tfrac{5}{4}e^{-10t}(1-10t) \text{ (volts)}}$$