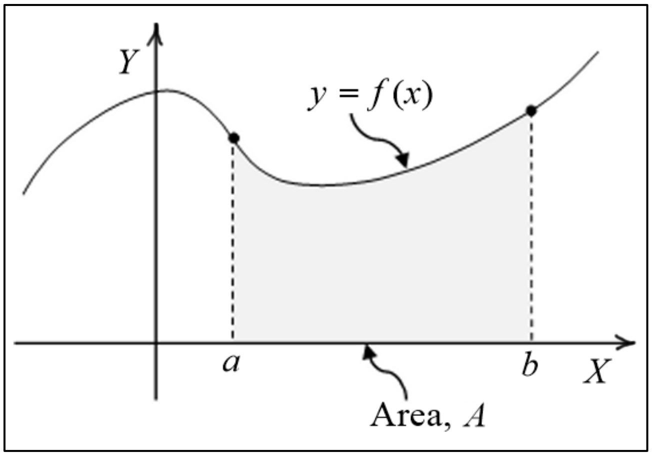
Elementary Engineering Mathematics
Introduction to the Integral of a Function

Definite Integral

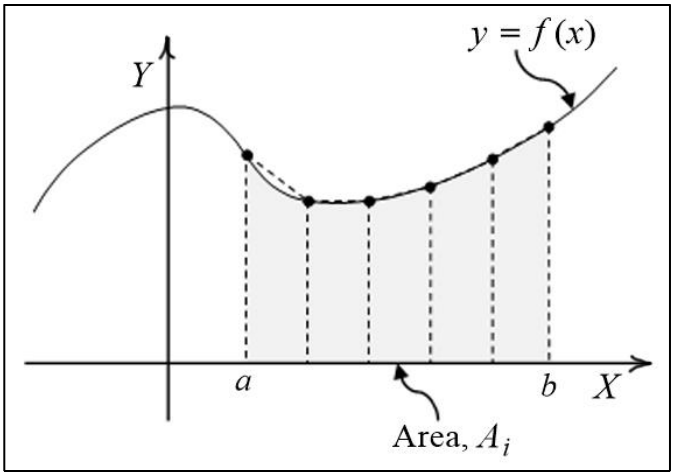
Consider a continuous function of a single variable $y = f(x)$. The ***integral*** of $f(x)$ from $x = a$ to $x = b$ is simply the ***area under the curve*** between those two points. The area is bounded on the underside by the *X*-axis. We write



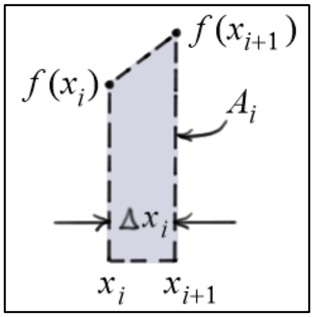
$$\boxed{A = \int_a^b f(x)dx} \qquad (1)$$

In this form, the integral is called a ***definite integral***. It is a ***number***, not a function. Later, we will define an ***indefinite integral*** which is itself a ***function*** of *x*.

One way to ***estimate*** the definite integral of a function is to break down the area into a finite number of ***trapezoids*** (as shown in the diagram) and ***sum*** the areas of all the trapezoids. As the increments Δx_i become smaller, Eq. (2) yields a more accurate estimate of the area *A*.



$$\boxed{\begin{aligned} A &\approx \sum_i A_i \\ &\approx \sum_i \left[\frac{f\left(x_{i+1}\right)+f\left(x_i\right)}{2}\right]\Delta x_i \\ &\approx \sum_i \left[\frac{f\left(x_{i+1}\right)+f\left(x_i\right)}{2}\right]\left(x_{i+1}-x_i\right) \end{aligned}} \qquad (2)$$



So far, we considered the function $f(x)$ and the increments Δx_i to be ***positive***. Consequently, the area *A* is ***positive***. If, however, $f(x)$ is ***negative*** over this range the area will be ***negative***. Clearly, if $f(x)$ takes on both ***positive*** and ***negative*** values, the resulting area could be ***positive***, ***negative***, or ***zero***.

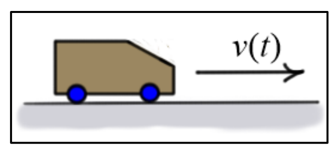
Given our current understanding of definite integrals, the following basic properties of integrals should seem reasonable.

	Property	Comment
1	$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	if function values have opposite sign, the areas will also
2	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	if increments have opposite sign, then areas will also
3	$\int_a^a f(x)\,dx = 0$	width of area is zero
4	$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	total area = the sum of the areas
5	$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$	α is a constant
6	$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$	integral of a sum = the sum of the integrals

Note on units: The units of an integral are the same as the units of $f(x) \times \Delta x$.

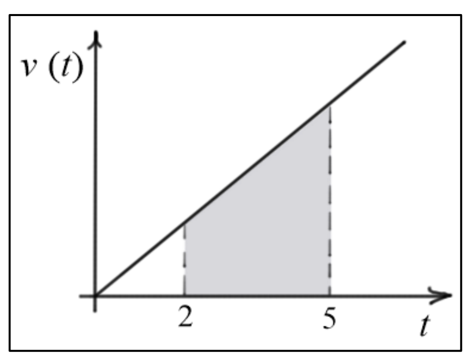
Example 1:

Given: The displacement of a car as it moves with velocity $v(t)$ from time t_1 to t_2 is the integral of $v(t)$ over that period of time.



$$\boxed{s = \int_{t_1}^{t_2} v(t)\,dt}$$

The displacement can be positive, negative, or zero depending on how the sign of $v(t)$ changes over the interval t_1 to t_2 .



Find: Assuming the car has velocity $v(t) = 7.5t$ (ft/s^2),

(a) find the displacement of the car from 2 to 5 seconds;

(b) find the total distance traveled from 2 to 5 seconds.

Solution:

(a) To calculate the shaded area, we can use a single trapezoid. The units of the result are the same as the units of $v(t)\times\Delta t \;\to\; (\text{ft/s})\times\text{s} \;\to\; (\text{ft})$.

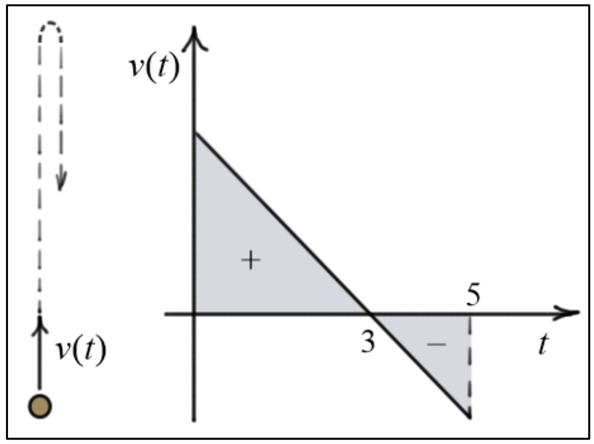
$$\boxed{s = A = (5-2)\big(v(2)+v(5)\big)/2 = 3(15+37.5)/2 = 78.75 \text{ (ft)}}$$

(b) The total distance traveled is also 78.75 (ft), because the velocity of the car is always positive from 2 to 5 seconds.

Example 2:

Given: The velocity of a ball for a certain period after it is thrown upward is

$$\boxed{v(t) = 96.6 - 32.2t \text{ (ft/s)}}$$



Find: (a) the vertical displacement of the ball from 0 to 5 seconds; and (b) the total distance traveled by the ball from 0 to 5 seconds.

Solution:

(a) The vertical displacement of the ball from 0 to 5 seconds is

$$\boxed{s = \tfrac{1}{2}(3)(96.6) - \tfrac{1}{2}(5-3)(64.4) = 80.5 \text{ (ft)}}$$

The downward movement of the ball is subtracted from the upward movement.

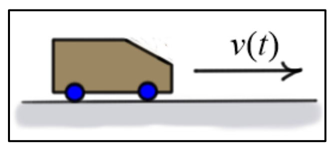
(b) The total distance traveled by the ball from 0 to 5 seconds is

$$\boxed{d = \tfrac{1}{2}(3)(96.6) + \tfrac{1}{2}(5-3)(64.4) = 209.3 \text{ (ft)}}$$

The upward and downward movements of the ball are summed.

Example 3:

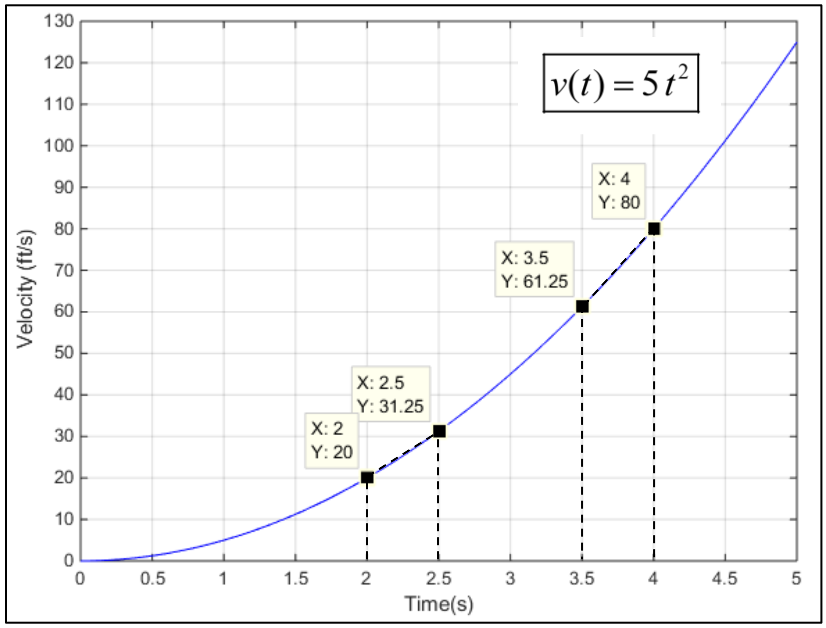
Given: The velocity of a car over the time interval from 0 to 5 seconds is $\boxed{v(t) = 5t^2 \text{ (ft/s)}}$.



Find: Estimate the distance traveled by the car from 2 to 5 seconds.

Solution:

Since the function is positive throughout the entire range of t, the total distance traveled is equal to the displacement. We can estimate the displacement by breaking up the area as shown in the diagram and the table below. ***Two*** of the areas are outlined on the plot with dashed lines.



Time, t	$f(t)$	Interval	f_{avg}	Δt
2	20	--		0.5
2.5	31.25	1	25.625	0.5
3	45	2	38.125	0.5
3.5	61.25	3	53.125	0.5
4	80	4	70.625	0.5
4.5	101.25	5	90.625	0.5
5	125	6	113.125	0.5
		Σ	391.25	

Since $\Delta t = 0.5$ (s) for all intervals, we have

$$s = A \approx \sum_{i=1}^{6} \left[\frac{f(x_{i+1}) + f(x_i)}{2} \right] \Delta t_i = \sum_{i=1}^{6} \left(f_{\text{avg}} \right)_i \Delta t_i = \Delta t \sum_{i=1}^{6} \left(f_{\text{avg}} \right)_i \approx 0.5 \times 391.25$$
$$\approx 195.625 \text{ (ft)}$$

We will see later that the ***actual displacement*** is 195 (ft). Our current result is in ***error*** by about 0.32%.