Elementary Engineering Mathematics
Properties of Functions

The following notes cover some of the ***basic concepts*** associated with functions of a ***single variable***. When studying more advanced mathematics (Calculus and beyond), it is important to understand these ***concepts*** and the ***terminology*** used to describe them.

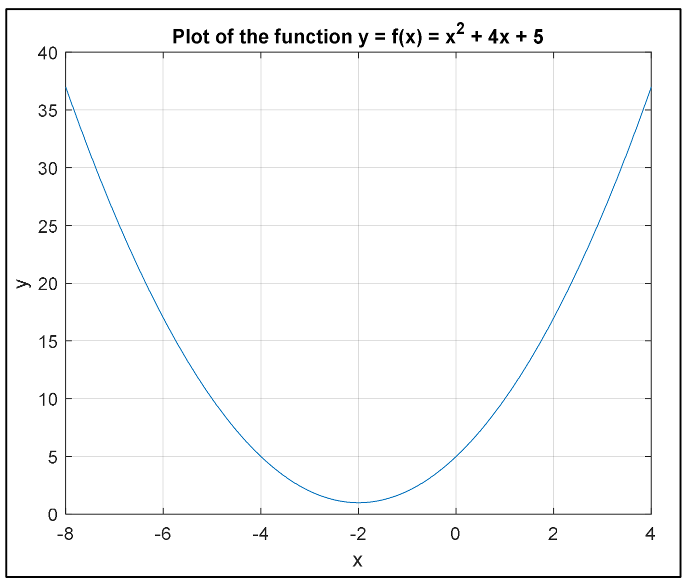
Functions of a Single Variable

For convenience, a ***set*** of ***numbers*** is often referred to by a ***single symbol*** (variable). For example, the symbol (or variable) x may be used to represent the set of ***all real numbers***, or it may be used to represent a ***smaller portion*** of them. The set of numbers that a variable x represents is called the ***range*** of the variable. If there is a ***process*** (or ***rule***) that can be used to calculate a ***single value*** of a variable y for ***each value*** in the range of the variable x, then the variable y is called a ***function*** of x. Symbolically we write $y = f(x)$ (we say, *f* of *x*). Here, x is called the ***independent variable***, y is called the ***dependent variable***, and $f(x)$ represents the ***process*** (or ***function***) used to calculate the value of variable y given a value of the variable x. The ***process*** used to calculate y is often based on an ***equation***.

Because a function can give only a ***single value*** for ***each value*** of the variable x, a ***vertical line*** on the ***plot*** of a ***function*** will ***cross*** the function only ***once***.

Domain and Range of a Function

The ***domain*** of a function is the ***complete set*** of numbers over which the function is ***defined***. The ***range*** of a function is the ***complete set*** of values the function takes on when applied to ***all*** the numbers in its domain.



The figure to the right shows the plot of the function $f(x) = x^2 + 4x + 5$ over the range of x values from $-8 \rightarrow 4$. Clearly, however, the value of the function is ***defined*** for ***any real number***. So, we write the ***domain*** of the function $f(x)$ as follows.

$$\text{domain}\left(f(x)\right) = \left(-\infty, \infty\right)$$

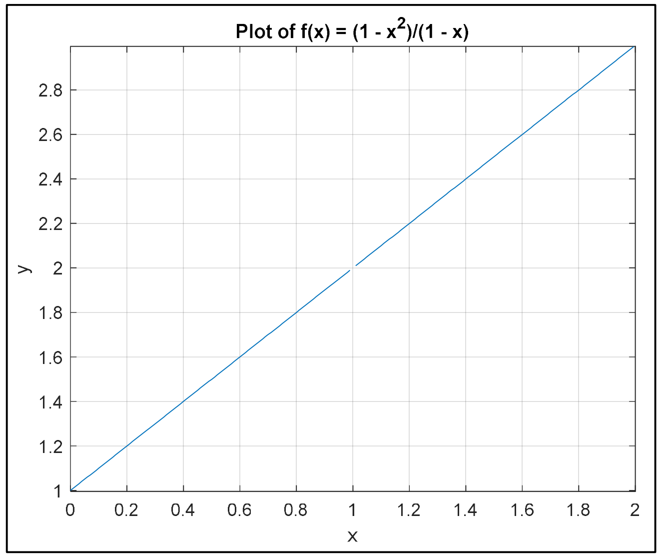
The ***parentheses*** are used to indicate that the range of values is up to but ***not including*** $-\infty$ or ∞.

The ***range*** of the function $f(x) = x^2 + 4x + 5$ varies from its ***minimum value*** of $f(-2) = 1$ to ever larger values as the value of x changes from $-2 \rightarrow -\infty$ and $-2 \rightarrow \infty$, so we write the range of $f(x)$ as follows.

$$\text{range}\left(f(x)\right) = [1, \infty)$$

The ***square bracket*** is used to indicate that the value of 1 is ***included*** in the range, and (as before) the parenthesis is used to indicate that the value range up to but not including ∞.

The figure to the right shows a plot of the function $f(x)=\left(1-x^2\right)/(1-x)$ over the range of x values from $0 \to 2$, ***excluding*** the value at $x=1$, because the function is ***not defined*** at that point. Clearly, the function takes on smaller and smaller values as $x \to -\infty$, and it takes on larger and larger values as $x \to \infty$. So, the ***domain*** of $f(x)$ is broken into ***two parts***: $(-\infty,1)$ and $(1,\infty)$, and the ***range*** is also broken into two parts: $(-\infty,2)$ and $(2,\infty)$. Again, parentheses are used to indicate the domain and range ***do not include*** their ***end points***.



Terminology: When ***intervals*** along the x-axis or y-axis ***do not include*** the ***end points***, they are called "***open intervals***", and when they ***do include*** the ***end points***, they are called "***closed intervals***".

Combining Functions

The simplest way to combine functions is by ***addition***, ***subtraction***, ***multiplication***, and ***division***. For functions that are defined over the ***same range*** of x values (i.e. they have the ***same domain***) these combinations are defined as follows.

$$\boxed{h(x)=(f \pm g)(x)=f(x) \pm g(x)} \quad \boxed{h(x)=(f \times g)(x)=f(x) \times g(x)} \quad \boxed{h(x)=(f/g)(x)=f(x)/g(x)}$$

Functions can also be combined using a process called ***composition***. If the ***range*** of a function $g(x)$ is ***within*** the ***domain*** of a function $f(x)$, then the ***composition*** of two functions written as $(f \circ g)(x)$ is defined as follows.

$$\boxed{(f \circ g)(x)=f(g(x))}$$

To understand the ***process*** of ***composition***, consider the following examples. Note that it is not uncommon for the composition to involve more than one variable. See Example #3.

Example #1: two functions of variable x

Given: $\boxed{f(x)=x^4+2x}$ $\boxed{g(x)=\sqrt{x}}$

Find: $\boxed{h(x)=f(g(x))}$

Solution:

$$\boxed{h(x)=f\left(\sqrt{x}\right)=\left(x^4+2x\right)_{x \to \sqrt{x}}=\left(\sqrt{x}\right)^4+2\sqrt{x}=x^2+2\sqrt{x}}$$

Example #2: two functions of variable x

Given: $\boxed{f(x)=1/(x+5)}$ $\boxed{g(x)=x^2+4x}$

Find: $\boxed{h(x)=f(g(x))}$

Solution:

$$\boxed{h(x)=f(g(x))=\left(\frac{1}{x+5}\right)_{x\to x^2+4x}=\frac{1}{x^2+4x+5}}$$

Example #3: one function of the angle θ with θ as a function of time t.

Given: $\boxed{f(\theta)=\sin(\theta)}$ $\boxed{\theta(t)=\omega t \quad (\omega \text{ is a constant})}$

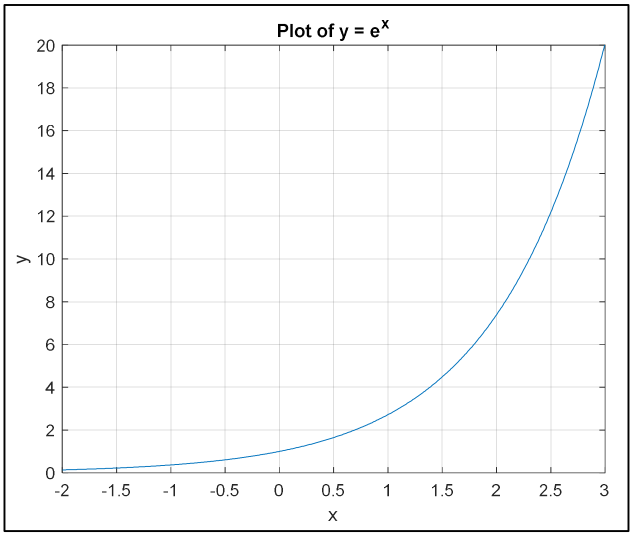
Find: $\boxed{h(t)=f(\theta(t))}$

Solution:

$$\boxed{h(t)=f(\theta(t))=(\sin(\theta))_{\theta\to\omega t}=\sin(\omega t)}$$

One-to-One Functions

A function is said to be a ***one-to-one function*** if the function value at each point in its domain is ***unique*** (no repeated values). For example, the function $f(x)=x^2$ is ***not*** a one-to-one function on the domain $(-\infty,\infty)$, because the value of the function is the ***same*** at $x=-a$ as it is at $x=a$. However, it ***is*** a one-to-one function on the domain $[0,\infty)$, because the square of any number $x\geq 0$ is unique. Clearly, it is also a one-to-one function on the domain $(-\infty,0]$.



The function $f(x)=e^x$ ***is*** a one-to-one function on the domain $(-\infty,\infty)$, because as the variable x ranges from $-\infty\to\infty$, the function increases from $0\to\infty$. ***No*** function values are repeated. The function $f(x)=e^x$ is plotted in the figure over the interval $[-2,3]$.

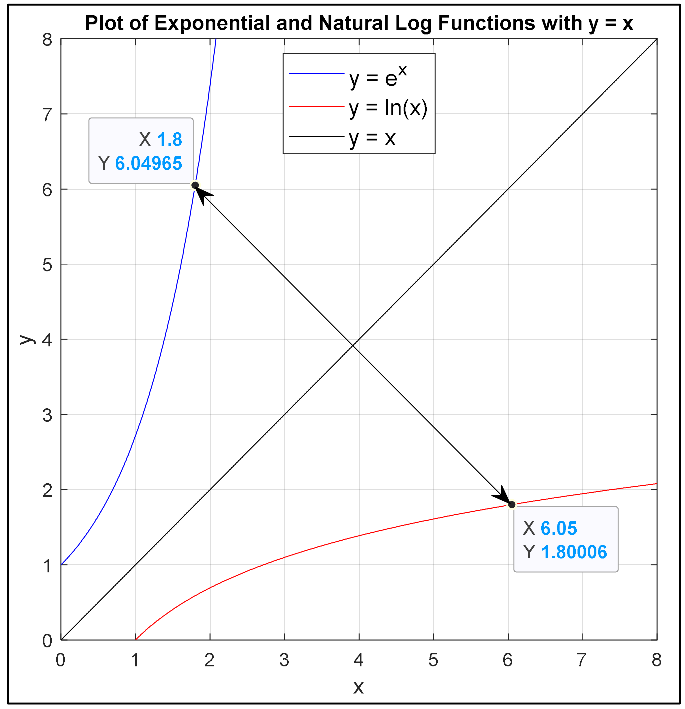
Inverse Functions

If $f(x)$ is a ***one-to-one function***, then $f^{-1}(x)$ (the ***inverse*** of $f(x)$) is that function such that $f^{-1}(f(x))=x$ at every point in the ***domain*** of $f(x)$. The function f converts x into $f(x)$, and the function f^{-1} converts $f(x)$ back into x. The functions $f(x)$ and $f^{-1}(x)$ are inverse of each other, so we can write the following.

$$f^{-1}\left(f(x)\right)=x \quad \text{and} \quad f\left(f^{-1}(x)\right)=x$$

Therefore, note that the ***domain*** of $f^{-1}(x)$ is the ***range*** of the function $f(x)$, and the ***domain*** of $f(x)$ is the ***range*** of $f^{-1}(x)$.

When plotted on the same graph, the functions $f(x)$ and $f^{-1}(x)$ are ***mirror images*** of each other on ***either side*** of the line $y=x$. The figure to the right shows a plot of the functions $y=e^{x}$ (blue line), $y=\log_e(x)$ (red line), and $y=x$ (black line). The ***exponential*** and ***natural logarithm*** functions are ***inverses*** of each other, so we can write the following.



$$e^{\log_e(x)}=x \quad \text{and} \quad \log_e\left(e^{x}\right)=x$$

These two statements are ***true*** at any *x*-value for which ***both functions*** are ***defined***.

Given the function $y=f(x)$ it is often ***easy*** to ***solve*** the equation for x to find $x=g(y)=f^{-1}(y)$. Consider the following examples.

Example #1:

Given: $f(x)=x^{3/5}$

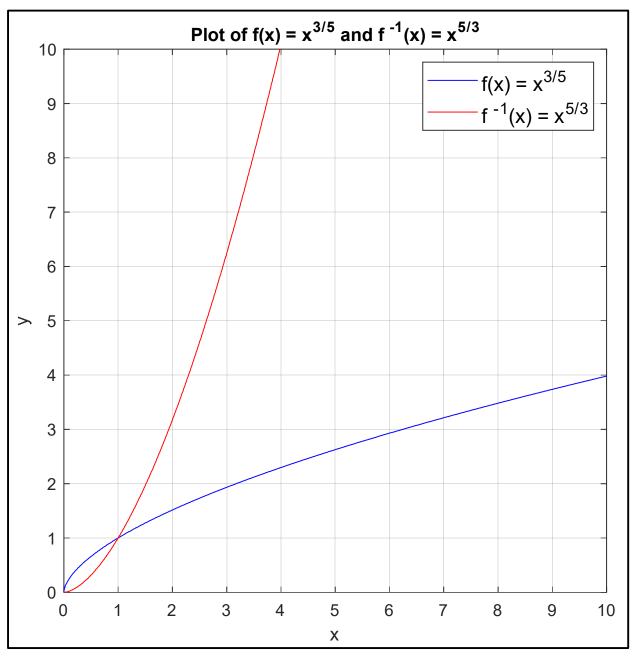
Find: $f^{-1}(x)$

Solution:

$$y=x^{3/5} \quad \Rightarrow y^{5/3}=(x^{3/5})^{5/3}=x \quad \Rightarrow \boxed{f^{-1}(x)=x^{5/3}}$$

Check: $\boxed{f^{-1}\left(f(x)\right)=\left(f(x)\right)^{5/3}=(x^{3/5})^{5/3}=x}$ ✓

The domain and range of both $f(x)$ and $f^{-1}(x)$ are $[0,\infty)$.



Example #2:

Given: $f(x) = 1 + x^5$

Find: $f^{-1}(x)$

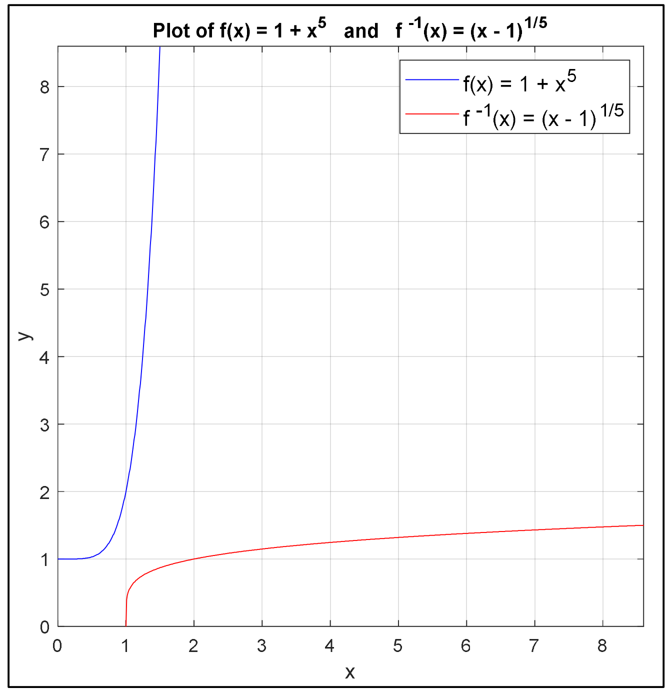
Solution:

$$y = 1 + x^5 \quad \Rightarrow x^5 = y - 1 \quad \Rightarrow x = (y-1)^{1/5}$$

$$\Rightarrow \boxed{f^{-1}(x) = (x-1)^{1/5}}$$

Check: $$\boxed{f^{-1}(f(x)) = (f(x)-1)^{1/5} = (1 + x^5 - 1)^{1/5} = (x^5)^{1/5} = x \checkmark}$$

The ***range*** of $f(x)$ is $[1, \infty)$ and the ***range*** of $f^{-1}(x)$ is $[0, \infty)$. Note that if $x < 1$, $f^{-1}(x)$ is ***undefined***.



Example #3:

Given: $f(x) = \frac{1}{1 + x^3}$

Find: $f^{-1}(x)$

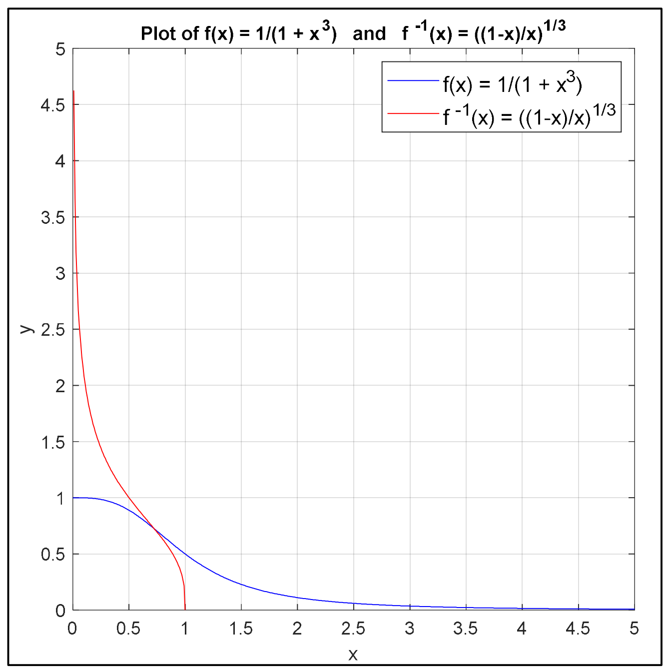
Solution:

$$y = \frac{1}{1+x^3} \quad \Rightarrow 1 + x^3 = \frac{1}{y} \quad \Rightarrow x^3 = \frac{1}{y} - 1 = \frac{1-y}{y}$$

$$\Rightarrow x = \left(\frac{1-y}{y}\right)^{1/3} \quad \Rightarrow \boxed{f^{-1}(x) = \left(\frac{1-x}{x}\right)^{1/3}}$$

Check:

$$\boxed{\begin{aligned} f^{-1}(f(x)) &= \left(\frac{1-f(x)}{f(x)}\right)^{1/3} = \left(\frac{1 - \frac{1}{1+x^3}}{\frac{1}{1+x^3}}\right)^{1/3} \\ &= \left(\frac{\frac{1+x^3-1}{1+x^3}}{\frac{1}{1+x^3}}\right)^{1/3} = \left(\left(\frac{x^3}{1+x^3}\right)(1+x^3)\right)^{1/3} \\ &= (x^3)^{1/3} = x \checkmark \end{aligned}}$$



The ***range*** of $f(x)$ is $(0,1]$ and the ***range*** of $f^{-1}(x)$ is $[0, \infty)$.

References:

D. Dwyer and M. Gruenwald, *Precalculus: MATH 118*, Thomson Wadsworth, 2006.

S. L. Salas and E. Hille, *Calculus: One and Several Variables*, Xerox College Publishing, 1971.