Limits of Functions at a Point

Reference:

S. L. Salas and E. Hille, *Calculus*: *One and Several Variables*, Xerox College Publishing, 1971.

Concept of a Limit:

The mathematical statement,

$$\boxed{\lim_{x\to c} f(x) = \ell} \tag{1}$$

means that as the value of x is made ***closer*** and ***closer*** to c, the value of the function $f(x)$ becomes ***closer*** and ***closer*** to ℓ. Specifically, the value of the difference $|f(x)-\ell|$ can be made smaller and smaller by decreasing the value of the difference $|x-c|$. Absolute values are taken of the differences "$f(x)-\ell$" and "$x-c$" to show that x can be ***increasing*** or ***decreasing*** toward c while $f(x)$ is be ***increasing*** or ***decreasing*** toward ℓ. For the ***limit*** to ***exist***, the function value must approach the value of ℓ from ***both sides*** of c.

How to Read Equation (1)**:**

Equation (1) should be read as follows:

The limit of function $f(x)$ as x approaches c is equal to ℓ.

Or,

As the value of x approaches c, the value of function $f(x)$ approaches ℓ.

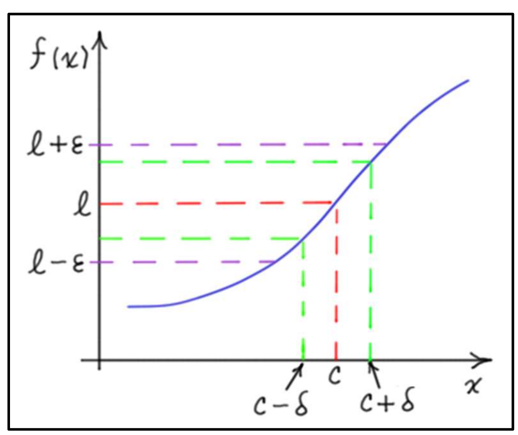
Formal Definition:

The $\lim_{x\to c} f(x) = \ell$ ***if and only if*** for each number $\varepsilon > 0$ ***there exists*** a number $\delta > 0$ such that if $0 < \lvert x-c\rvert < \delta$, then $\lvert f(x)-\ell\rvert < \varepsilon$.	(2)

Note that the inequality $0 < |x-c| < \delta$ requires the value of x lie within the band of width 2δ from $c-\delta$ to $c+\delta$, and the inequality $|f(x)-\ell| < \varepsilon$ requires the value of $f(x)$ lie within the band of width 2ε from $\ell-\varepsilon$ to $\ell+\varepsilon$.

Illustration:

The figure shows a function $f(x)$ as it passes through the region around $x = c$. The ***red*** dashed lines indicate the values of c and ℓ; the ***green*** dashed lines indicate the values of $c-\delta$, $c+\delta$, and the function values at these points; and the ***purple*** lines indicate the ε band around the value of ℓ. If $\lim_{x\to c} f(x) = \ell$, then the ***function values*** at $c-\delta$ and $c+\delta$ must fall ***within*** the ε ***band***, and as the ***value*** of ε is decreased, the ***value*** of δ can be decreased as well to ensure the function values remain in the ε band. Finally, as the value of $\delta \to 0$, the function values at $c-\delta$ and $c+\delta$ must approach ℓ.



Notes:

1. For the function $f(x)$ to have a limit at $x = c$, it needs to be defined ***close to*** $x = c$, but it need ***not be*** defined at $x = c$.
2. To check the limit of the function at $x = c$, the ***relative values*** of the parameters ε and δ for any given value of c can related by $f(x)$. Once the value of ***either parameter*** is ***chosen***, an ***acceptable value*** of the ***second parameter*** can be found using $f(x)$ and c.
3. In each of the examples that follow, to verify the limit of the function at some point $x = c$, the limit is checked both from ***above*** and ***below*** $x = c$. These limits are often written as follows.

   Limit from below: $\boxed{\lim_{x \uparrow c} f(x)}$ or $\boxed{\lim_{x \to c^-} f(x)}$

   Limit from above: $\boxed{\lim_{x \downarrow c} f(x)}$ or $\boxed{\lim_{x \to c^+} f(x)}$

Useful properties of limits:

1. If $\boxed{\lim_{x \to c} f(x) = \ell}$ and $\boxed{\lim_{x \to c} g(x) = m}$, then

   a) $\boxed{\lim_{x \to c}\big(f(x) + g(x)\big) = \ell + m}$ b) $\boxed{\lim_{x \to c} \alpha f(x) = \alpha\ell \text{ (for any real } \alpha)}$ c) $\boxed{\lim_{x \to c}\big(f(x)g(x)\big) = \ell m}$

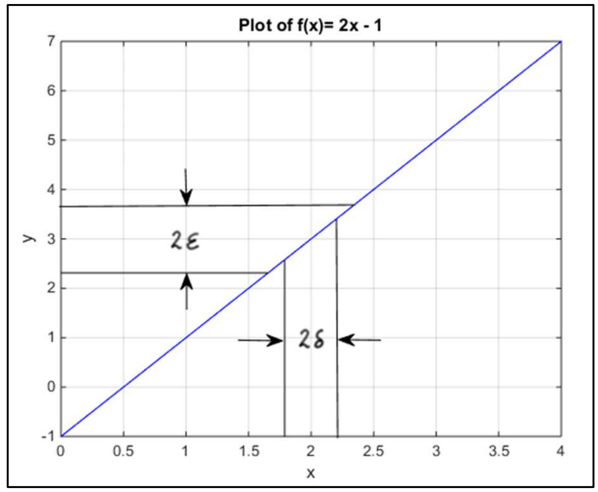
2. If $\boxed{\lim_{x \to c} f(x) = \ell}$ and $\boxed{\lim_{x \to c} g(x) = m \neq 0}$, then $\boxed{\lim_{x \to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\ell}{m}}$
3. For any polynomial $P(x)$, $\boxed{\lim_{x \to c} P(x) = P(c)}$
4. For polynomials $P(x)$ and $Q(x)$:

   If $\boxed{\lim_{x \to c} P(x) = P(c)}$ and $\boxed{\lim_{x \to c} Q(x) = Q(c) \neq 0}$, then $\boxed{\lim_{x \to c}\left(\frac{P(x)}{Q(x)}\right) = \frac{P(c)}{Q(c)}}$

5. If $\boxed{\lim_{x \to c} f(x) = \ell \neq 0}$ and $\boxed{\lim_{x \to c} g(x) = 0}$, then $\boxed{\lim_{x \to c}\left(\frac{f(x)}{g(x)}\right) \text{ does not exist}}$

Examples:

Example #1:



Verify the following limit: $\boxed{\lim_{x\to 2} f(x) \triangleq \lim_{x\to 2}(2x-1) = 3}$

Solution:

1. Set the range for x: $(2-\delta) < x < (2+\delta)$
2. Set the range for $f(x)$: $(3-\varepsilon) < f(x) < (3+\varepsilon)$

   (the function is increasing)
3. Calculate the function values at the ***end points*** of the x-range:

   $$f(x-\delta)\big|_{x=2} = \big(2(x-\delta)-1\big)_{x=2} = \big((2x-1)-2\delta\big)_{x=2} = 3-2\delta$$

   $$f(x+\delta)\big|_{x=2} = \big(2(x+\delta)-1\big)_{x=2} = \big((2x-1)+2\delta\big)_{x=2} = 3+2\delta$$
4. Comparing the results from step 3 to the limits set in step 2, we require

   $\boxed{3-2\delta > 3-\varepsilon}$ and $\boxed{3+2\delta < 3+\varepsilon}$
5. First condition:

   $3-2\delta > 3-\varepsilon \quad \Rightarrow -2\delta > \varepsilon \quad \Rightarrow 2\delta < \varepsilon \quad \boxed{\delta < \tfrac{1}{2}\varepsilon}$
6. Second condition:

   $3+2\delta < 3+\varepsilon \quad \Rightarrow 2\delta < \varepsilon \quad \Rightarrow \boxed{\delta < \tfrac{1}{2}\varepsilon}$
7. Check: Let $\boxed{\varepsilon = 0.1}$ and $\boxed{\delta = 0.04 < \tfrac{1}{2}\varepsilon = 0.05}$

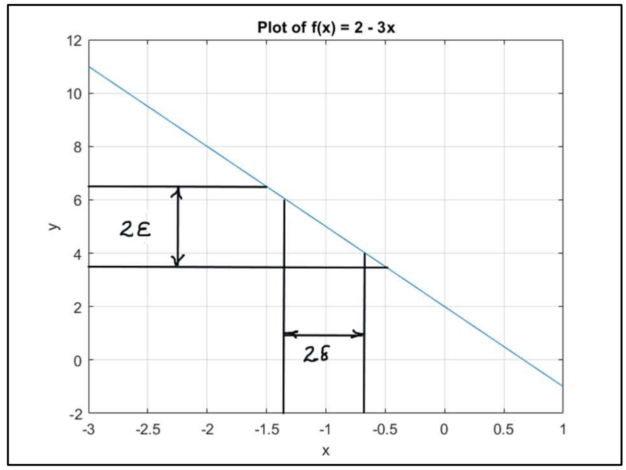
   Note: $3-\varepsilon = 2.9$ and $3+\varepsilon = 3.1$

   $\boxed{f(x-0.04)\big|_{x=2} = \big(2(x-0.04)-1\big)_{x=2} = 2.92 > 3-\varepsilon = 2.9}$ ✓

   $\boxed{f(x+0.04)\big|_{x=2} = \big(2(x+0.04)-1\big)_{x=2} = 3.08 < 3+\varepsilon = 3.1}$ ✓
8. As $\varepsilon \to 0$: $\left(\delta < \tfrac{1}{2}\varepsilon\right) \to 0$, $f(x-\delta)\big|_{x=2} \to f(x)\big|_{x=2} = 3$, and $f(x+\delta)\big|_{x=2} \to f(x)\big|_{x=2} = 3$
9. These results are ***consistent*** with the graph of the function.

Example #2:

Verify the following limit: $\boxed{\lim_{x\to -1} f(x) \triangleq \lim_{x\to -1}\left(2-3x\right)=5}$



Solution:

1. Set the range for x: $\left(-1-\delta\right) < x < \left(-1+\delta\right)$
2. Set the range for $f(x)$: $\left(5+\varepsilon\right) > f(x) > \left(5-\varepsilon\right)$

   (the function is decreasing)
3. Calculate the function values at the end points of the x-range:

   $$f(x-\delta)\big|_{x=-1} = \left(2-3\left(x-\delta\right)\right)_{x=-1} = \left(\left(2-3x\right)+3\delta\right)_{x=-1} = 5+3\delta$$

   $$f(x+\delta)\big|_{x=-1} = \left(2-3\left(x+\delta\right)\right)_{x=-1} = \left(\left(2-3x\right)-3\delta\right)_{x=-1} = 5-3\delta$$
4. Comparing the results from step 3 to the limits set in step 2, we require

   $\boxed{5+3\delta < 5+\varepsilon}$ and $\boxed{5-3\delta > 5-\varepsilon}$
5. First condition:

   $5+3\delta < 5+\varepsilon \quad \Rightarrow 3\delta < \varepsilon \quad \boxed{\delta < \tfrac{1}{3}\varepsilon}$
6. Second condition:

   $5-3\delta > 5-\varepsilon \quad \Rightarrow -3\delta > -\varepsilon \quad \Rightarrow 3\delta < \varepsilon \quad \Rightarrow \boxed{\delta < \tfrac{1}{3}\varepsilon}$
7. Check: Let $\boxed{\varepsilon = 0.1}$ and $\boxed{\delta = 0.03 < \tfrac{1}{3}\varepsilon = 0.0\overline{33}}$

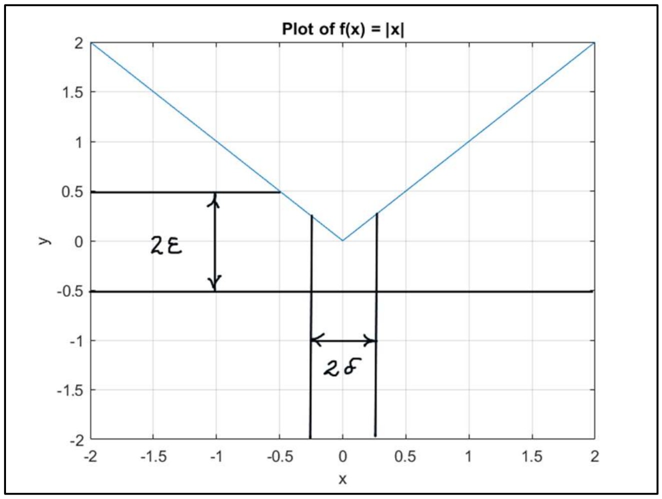
   Note: $5-\varepsilon = 4.9$ and $5+\varepsilon = 5.1$

   $\boxed{f(x-0.03)\big|_{x=-1} = \left(2-3(x-0.03)\right)_{x=-1} = 5.09 < 5+\varepsilon = 5.1}$ ✓

   $\boxed{f(x+0.03)\big|_{x=-1} = \left(2-3(x+0.03)\right)_{x=-1} = 4.91 > 5-\varepsilon = 4.9}$ ✓
5. As $\varepsilon \to 0$: $\left(\delta < \tfrac{1}{3}\varepsilon\right) \to 0$, $f(x-\delta)\big|_{x=-1} \to f(x)\big|_{x=-1} = 5$, and $f(x+\delta)\big|_{x=-1} \to f(x)\big|_{x=-1} = 5$
6. These results are ***consistent*** with the graph of the function.

Example #3:

Verify the following limit: $\boxed{\lim_{x\to 0} f(x) \triangleq \lim_{x\to 0} |x| = 0}$



Solution:

1. Set the range for x: $-\delta < x < \delta$
2. Set the range for $f(x)$: $-\varepsilon < f(x) < \varepsilon$

   (the function bottoms out at zero)
3. Calculate the function values at the end points of the x-range:

   $f(x-\delta)\big|_{x=0} = (|x|)_{x=-\delta} = \delta$

   $f(x+\delta)\big|_{x=0} = (|x|)_{x=\delta} = \delta$
4. Comparing the results from step 3 to the limits set in step 2, we require

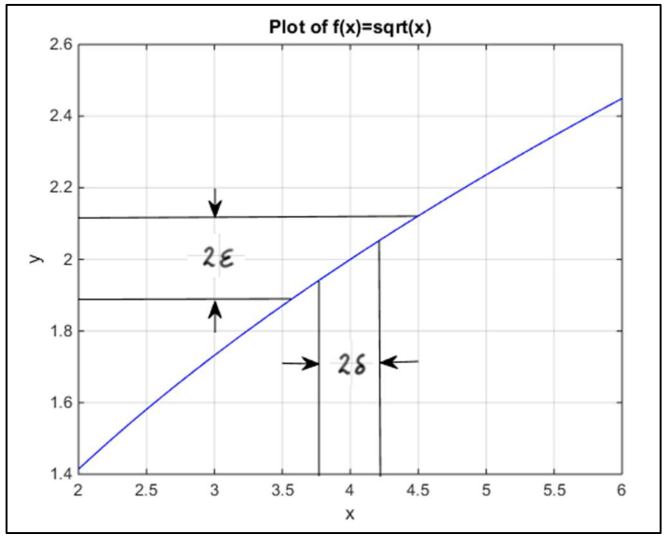
   $\boxed{\delta > -\varepsilon}$ and $\boxed{\delta < \varepsilon}$
5. First condition:
   Both ε and δ are ***positive***, so this condition is satisfied. No new information here.
6. Second condition:
   $\boxed{\delta < \varepsilon}$
7. Check: Let $\boxed{\varepsilon = 0.1}$ and $\boxed{\delta = 0.09 < \varepsilon}$

   $\boxed{f(x+0.09)\big|_{x=0} = (|x|)_{x=0.09} = 0.09 < \varepsilon}$ ✓

   $\boxed{f(x-0.09)\big|_{x=0} = (|x|)_{x=-0.09} = 0.09 < \varepsilon}$ ✓
5. As $\varepsilon \to 0$: $(\delta < \varepsilon) \to 0$, $f(x-\delta)\big|_{x=0} \to f(x)\big|_{x=0} = 0$, and $f(x+\delta)\big|_{x=0} \to f(x)\big|_{x=0} = 0$
6. These results are ***consistent*** with the graph of the function.

Example #4:

Verify the following limit: $\lim_{x\to 4} f(x) \triangleq \lim_{x\to 4} \sqrt{x} = 2$



Solution:

1. Set the range for x: $\left(4-\delta\right) < x < \left(4+\delta\right)$
2. Set the range for $f(x)$: $\left(2-\varepsilon\right) < f(x) < \left(2+\varepsilon\right)$
3. Calculate the function values at the ***end points*** of the x-range:

$$f(x-\delta)\big|_{x=4} = \left(\sqrt{x-\delta}\right)_{x=4} = \sqrt{4-\delta}$$
$$f(x+\delta)\big|_{x=4} = \left(\sqrt{x+\delta}\right)_{x=4} = \sqrt{4+\delta}$$

4. Comparing the results from step 3 to the limits set in step 2, we require

$$\boxed{\sqrt{4-\delta} > 2-\varepsilon} \quad \text{and} \quad \boxed{\sqrt{4+\delta} < 2+\varepsilon}$$

5. First condition:

Note first that $4-\delta > 1$ and $2-\varepsilon > 1$, so both sides can be squared without changing the inequality.

$$\left(\sqrt{4-\delta}\right)^2 > \left(2-\varepsilon\right)^2 \quad \Rightarrow \quad 4-\delta > 4-4\varepsilon+\varepsilon^2 \quad \Rightarrow \quad -\delta > -4\varepsilon+\varepsilon^2 \quad \Rightarrow \boxed{\delta < 4\varepsilon-\varepsilon^2}$$

6. Second condition:

Note first that $4+\delta > 1$ and $2+\varepsilon > 1$, so both sides can be squared without changing the inequality.

$$\left(\sqrt{4+\delta}\right)^2 < \left(2+\varepsilon\right)^2 \quad \Rightarrow \quad 4+\delta < 4+4\varepsilon+\varepsilon^2 \quad \Rightarrow \quad \boxed{\delta < 4\varepsilon+\varepsilon^2}$$

7. The ***first condition*** that $\boxed{\delta < 4\varepsilon-\varepsilon^2}$ should be used as it yields a ***smaller value*** for δ.

8. Check: Let $\boxed{\varepsilon = 0.1}$ and $\boxed{\delta = 0.38 < 4\varepsilon-\varepsilon^2 = 0.39}$

Note: $2-\varepsilon = 1.9$ and $2+\varepsilon = 2.1$

$$\boxed{f(x-0.38)\big|_{x=4} = \left(\sqrt{x-0.38}\right)_{x=4} = 1.90263 > 2-\varepsilon = 1.9} \checkmark$$
$$\boxed{f(x+0.38)\big|_{x=4} = \left(\sqrt{x+0.38}\right)_{x=4} = 2.0928 < 2+\varepsilon = 2.1} \checkmark$$

9. As $\varepsilon \to 0$: $\left(\delta < 4\varepsilon-\varepsilon^2\right) \to 0$ and $f(x-\delta)\big|_{x=4} \to f(x)\big|_{x=4} = 2$

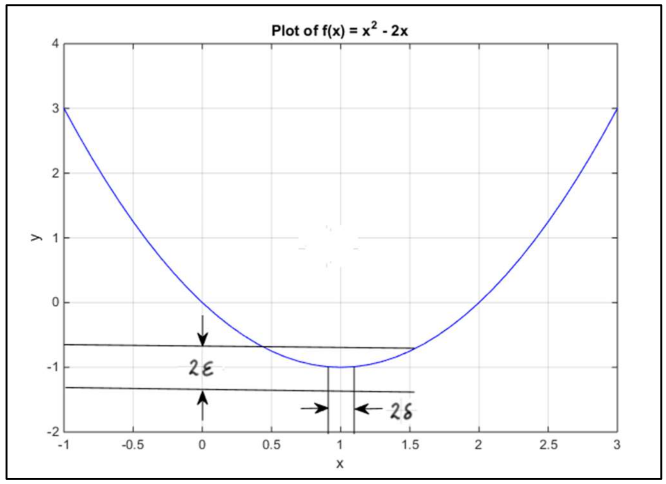
As $\varepsilon \to 0$: $\left(\delta < 4\varepsilon+\varepsilon^2\right) \to 0$ and $f(x+\delta)\big|_{x=4} \to f(x)\big|_{x=4} = 2$

10. These results are ***consistent*** with the graph of the function.

Example #5:

Verify the following limit:

$$\lim_{x\to 1} f(x) \triangleq \lim_{x\to 1}\left(x^2-2x\right) = -1$$



Solution:

1. Set the range for x: $(1-\delta) < x < (1+\delta)$
2. Set the range for $f(x)$: $(-1-\varepsilon) < f(x) < (-1+\varepsilon)$
3. Calculate the function values at the ***end points*** of the *x*-range:

$$f(x-\delta)\big|_{x=1} = \left((x-\delta)^2 - 2(x-\delta)\right)_{x=1} = (1-\delta)^2 - 2(1-\delta) = \left(1-2\delta+\delta^2\right) + (2\delta-2) = \delta^2 - 1$$

$$f(x+\delta)\big|_{x=1} = \left((x+\delta)^2 - 2(x+\delta)\right)_{x=1} = (1+\delta)^2 - 2(1+\delta) = \left(1+2\delta+\delta^2\right) - (2\delta+2) = \delta^2 - 1$$

4. Comparing the results from step 3 to the limits set in step 2, we require

   $\boxed{\delta^2 - 1 > -1-\varepsilon}$ and $\boxed{\delta^2 - 1 < -1+\varepsilon}$

5. First condition for small δ and ε:

   Note first that $\delta^2 - 1 < 0$ and $-1-\varepsilon < 0$. Then,

   $$\delta^2 - 1 > -1-\varepsilon \;\Rightarrow\; \delta^2 > -\varepsilon \;\Rightarrow\; \boxed{\delta > 0}$$

   This result may seem odd, but the function increases in value for $x < 1$ and never crosses the lower limit of $-1-\varepsilon$. See the graph of the function.

6. Second condition for small δ and ε:

   Note first that $\delta^2 - 1 < 0$ and $-1+\varepsilon < 0$. Then,

   $$\delta^2 - 1 < -1+\varepsilon \;\Rightarrow\; \delta^2 < \varepsilon \;\Rightarrow\; \boxed{\delta < \sqrt{\varepsilon}}$$

7. The ***first condition*** that $\boxed{\delta > 0}$ adds ***no new information*** as δ is required to be greater than zero, so we use the ***second condition*** $\delta < \sqrt{\varepsilon}$.

8. Check: Let $\boxed{\varepsilon = 0.1}$ and $\boxed{\delta = 0.3 < \sqrt{\varepsilon} = 0.31623}$

   Note: $-1-\varepsilon = -1.1$ and $-1+\varepsilon = -0.9$

   $$\boxed{f(x-0.3)\big|_{x=1} = \left((x-0.3)^2 - 2(x-0.3)\right)_{x=1} = -0.91 > -1+\varepsilon} \checkmark$$

   $$\boxed{f(x+0.3)\big|_{x=1} = \left((x+0.3)^2 - 2(x+0.3)\right)_{x=1} = -0.91 < -1+\varepsilon} \checkmark$$

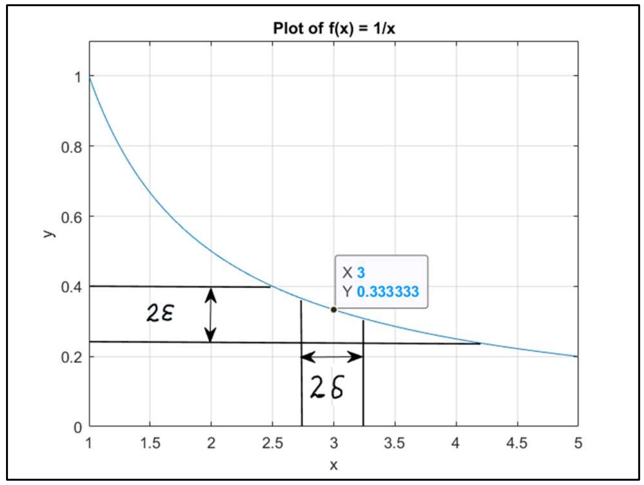
9. As $\varepsilon \to 0$: $\left(\delta < \sqrt{\varepsilon}\right) \to 0$ and $f(x-\delta)\big|_{x=1} \to f(x)\big|_{x=1} = -1$

   As $\varepsilon \to 0$: $\left(\delta < \sqrt{\varepsilon}\right) \to 0$ and $f(x+\delta)\big|_{x=1} \to f(x)\big|_{x=1} = -1$

10. These results are ***consistent*** with the graph of the function.

Example #6:

Verify the following limit: $\lim_{x \to 3} f(x) \triangleq \lim_{x \to 3} \left(\frac{1}{x}\right) = \frac{1}{3}$



Solution:

1. Set the range for x: $(3-\delta) < x < (3+\delta)$

2. Set the range for $f(x)$: $\left(\frac{1}{3}+\varepsilon\right) > f(x) > \left(\frac{1}{3}-\varepsilon\right)$

   (the function is decreasing)

3. Calculate the function values at the ***end points*** of the x-range:

   $$f(x-\delta)\Big|_{x=3} = \left(\frac{1}{x-\delta}\right)_{x=3} = \frac{1}{3-\delta}$$
   $$f(x+\delta)\Big|_{x=3} == \left(\frac{1}{x+\delta}\right)_{x=3} = \frac{1}{3+\delta}$$

4. Comparing the results from step 3 to the limits set in step 2, we require

   $\boxed{\frac{1}{3-\delta} < \left(\frac{1}{3}+\varepsilon\right)}$ and $\boxed{\frac{1}{3+\delta} > \left(\frac{1}{3}-\varepsilon\right)}$

5. First condition: Note first that for small ε and δ that $3-\delta > 0$ and $\frac{1}{3}+\varepsilon > 0$

   $\frac{1}{3-\delta} < \left(\frac{1}{3}+\varepsilon\right) = \frac{1+3\varepsilon}{3}$ $\Rightarrow 3-\delta > \frac{3}{1+3\varepsilon}$ $\Rightarrow \delta - 3 < -\frac{3}{1+3\varepsilon}$ $\Rightarrow \delta < 3 - \frac{3}{1+3\varepsilon}$ $\delta < \frac{3(1+3\varepsilon)-3}{1+3\varepsilon}$

   $\boxed{\delta < \frac{9\varepsilon}{1+3\varepsilon}}$

6. Second condition:

   $\frac{1}{3+\delta} > \frac{1}{3} - \varepsilon$ $\Rightarrow \frac{1}{3+\delta} > \frac{1-3\varepsilon}{3}$ $\Rightarrow 3+\delta < \frac{3}{1-3\varepsilon}$ $\Rightarrow \delta < \frac{3}{1-3\varepsilon} - 3 = \frac{3-3(1-3\varepsilon)}{1-3\varepsilon}$ $\Rightarrow \boxed{\delta < \frac{9\varepsilon}{1-3\varepsilon}}$

7. The first condition $\boxed{\delta < \frac{9\varepsilon}{1+3\varepsilon}}$ should be used as it yields a ***smaller value*** for δ.

8. Check: Let $\boxed{\varepsilon = 0.1}$ then

   $$\boxed{\delta < \left[\frac{9\varepsilon}{1+3\varepsilon}\right]_{\varepsilon=0.1} = 0.6923}$$

   Using $\delta = 0.65$

   $$\boxed{f(x-0.65)\Big|_{x=3} = \left(\frac{1}{x-0.65}\right)_{x=3} = 0.4255 < \frac{1}{3}+\varepsilon = 0.4\overline{33}}$$ ✓

   $$\boxed{f(x+0.65)\Big|_{x=3} = \left(\frac{1}{x+0.65}\right)_{x=3} = 0.2740 > \frac{1}{3}-\varepsilon = 0.2\overline{33}}$$ ✓

9. As $\varepsilon \to 0$: $\left(\delta < \frac{9\varepsilon}{1+3\varepsilon}\right) \to 0$ and $f(x-\delta)\Big|_{x=3} \to f(x)\Big|_{x=3} = \frac{1}{3}$

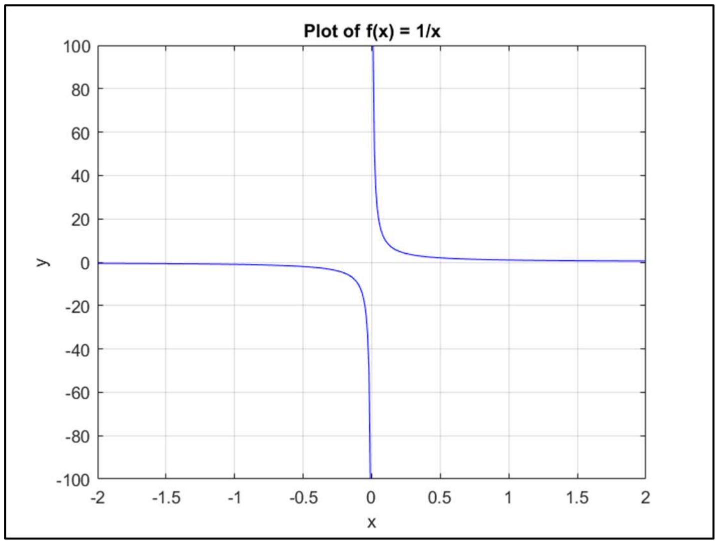
   As $\varepsilon \to 0$: $\left(\delta < \frac{9\varepsilon}{1-3\varepsilon}\right) \to 0$ and $f(x-\delta)\Big|_{x=3} \to f(x)\Big|_{x=3} = \frac{1}{3}$

10. These results are ***consistent*** with the graph of the function.

Example #7:

Verify that the following limit ***does not exist*** by plotting the function $f(x) = \frac{1}{x}$.

$$\lim_{x \to 0} f(x) \triangleq \lim_{x \to 0} \left(\frac{1}{x} \right)$$



Solution:

The figure shows a plot of the function $f(x) = \frac{1}{x}$ over the domain $[-2, 0)$ and $(0, +2]$. For the limit of the function at $x = 0$ to exist, the function need not be defined, but the function must approach the same value from above ($x > 0$) as it does from below ($x < 0$).

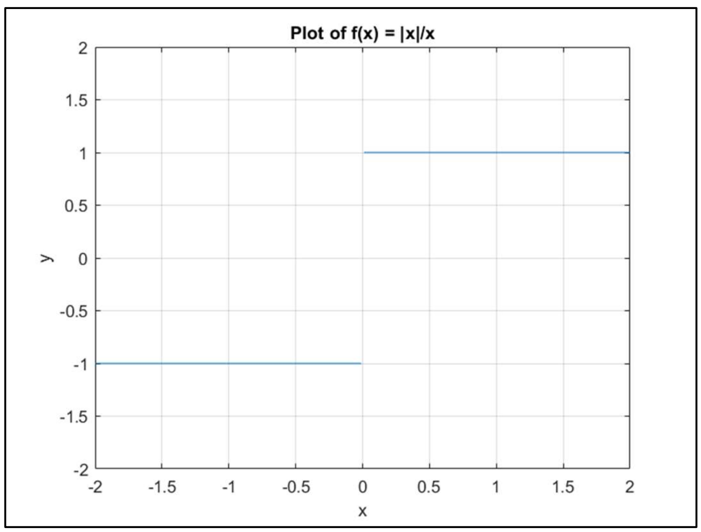
The value of the function $f(x) = \frac{1}{x}$ is not defined at $x = 0$. The function is singular at this point. Clearly, when $x < 0$, the function takes on larger and larger ***negative*** values as $x \to 0$, but when $x > 0$, the function takes on larger and larger ***positive*** values as $x \to 0$. Clearly the function does not approach the same value on either side of the singularity, so $\lim_{x \to 0} \left(\frac{1}{x} \right)$ ***does not exist***.

This is an example of property 5 listed above.

Example #8:

Verify that the following limit ***does not exist*** by plotting the function $f(x) = \frac{|x|}{x}$.

$$\lim_{x \to 0} f(x) \triangleq \lim_{x \to 0} \left(\frac{|x|}{x} \right)$$



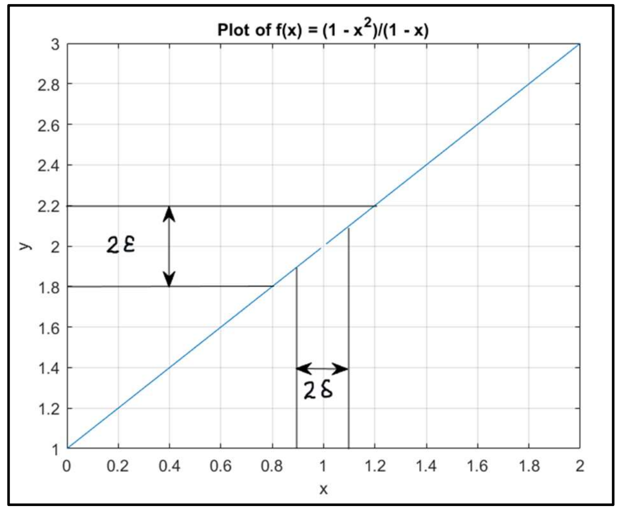
Solution:

The figure shows a plot of the function $f(x) = \frac{|x|}{x}$ over the domain $[-2, 0)$ and $(0, +2]$. The function is ***undefined*** (singular) at $x = 0$, and the left and right branches approach different values as $x \to 0$. The left branch approaches -1 while the right branch approaches $+1$. So, $\lim_{x \to 0} \left(\frac{|x|}{x} \right)$ ***does not exist***.

Example #9:

Verify the following limit:

$$\lim_{x\to 1} f(x) \triangleq \lim_{x\to 1}\left(\frac{1-x^2}{1-x}\right) = 2$$



Solution: The function is singular at $x = 1$.

1. Set the range for x: $(1-\delta) < x < (1+\delta)$
2. Set the range for $f(x)$: $(2-\varepsilon) < f(x) < (2+\varepsilon)$
3. Calculate the function values at the ***end points*** of the x-range:

$$f(x-\delta)_{x=1} = \frac{1-(1-\delta)^2}{1-(1-\delta)} = \frac{1-\left(1-2\delta+\delta^2\right)}{\delta} = \frac{2\delta-\delta^2}{\delta} = 2-\delta$$

$$f(x+\delta)_{x=1} = \frac{1-(1+\delta)^2}{1-(1+\delta)} = \frac{1-\left(1+2\delta+\delta^2\right)}{-\delta} = \frac{-2\delta-\delta^2}{-\delta} = 2+\delta$$

4. Comparing the results from step 3 to the limits set in step 2, we require

$\boxed{2-\delta > 2-\varepsilon}$ and $\boxed{2+\delta < 2+\varepsilon}$

5. Conditions 1 and 2:

$$2-\delta > 2-\varepsilon \quad \Rightarrow -\delta > -\varepsilon \quad \Rightarrow \boxed{\delta < \varepsilon}$$
$$2+\delta < 2+\varepsilon \quad \Rightarrow \boxed{\delta < \varepsilon}$$

6. Check: Let $\boxed{\varepsilon = 0.1}$ and $\boxed{\delta = 0.09 < \varepsilon = 0.1}$

$$\boxed{f(x-0.09)_{x=1} = \frac{1-(1-0.09)^2}{1-(1-0.09)} = 1.91 > (2-\varepsilon)_{\varepsilon=0.1} = 1.9} \checkmark$$

$$\boxed{f(x+0.09)_{x=1} = \frac{1-(1+0.09)^2}{1-(1+0.09)} = 2.09 < (2+\varepsilon)_{\varepsilon=0.1} = 2.1} \checkmark$$

7. Note that the function $f(x) = \frac{1-x^2}{1-x}$ is undefined at, however the left and right branches approach the same value. That is, they both approach the value of 2.

As $\varepsilon \to 0$: $(\delta < \varepsilon) \to 0$ and $f(x-\delta)\big|_{x=1} = 2-\delta \quad \to \lim_{\delta\to 0}(2-\delta) = 2$

As $\varepsilon \to 0$: $(\delta < \varepsilon) \to 0$ and $f(x+\delta)\big|_{x=1} = 2+\delta \quad \to \lim_{\delta\to 0}(2+\delta) = 2$

8. These results are ***consistent*** with the graph of the function. Note that the limits from the ***left*** and ***right*** both approach a value of 2 even though the function is ***not defined*** at $x = 1$.