Limits and Continuity

Continuity at a Point

A function $f(x)$ is said to be ***continuous*** at $x = c$, if and only if, $\boxed{\lim_{x \to c} f(x) = f(c)}$. So, not only does the limit have to ***exist***, but it also must be ***equal*** to the ***function value*** at that point. The first six examples in the previous notes on "Limits of Functions at a Point" were examples of ***continuous functions***. The last three examples were examples of ***discontinuous functions***. Two of the discontinuous functions had ***no limit*** at the point in question, and one had a limit, but the limit was ***not equal*** to the ***function value*** at that point (the function value was not defined). Both continuous and discontinuous functions are found in engineering applications.

Using the definition of the limit of a function at a point, the definition of continuity at a point can be written more formally as follows.

A function $f(x)$ is said to be continuous at $x = c$, if and only if, for each number $\varepsilon > 0$ ***there exists*** a number $\delta > 0$ such that if $0 < \lvert x - c \rvert < \delta$, then $\lvert f(x) - f(c) \rvert < \varepsilon$

This statement is the same as that for the limit of a function at a point, but it also requires the limit to be the function value. Less formally, a function is continuous at $x = c$ if and only if for x close to c, $f(x)$ is close to $f(c)$.

Useful properties of Continuity:

Using the ***properties*** of ***limits*** given in the previous notes, the following statements can be made about continuous functions.

1. If the functions $f(x)$ and $g(x)$ are ***continuous*** at $x = c$, then

   a) The function $h(x) = f(x) + g(x)$ is ***continuous*** at $x = c$.

   b) The function $h(x) = \alpha f(x)$ is ***continuous*** at $x = c$. (α is any constant)

   c) The function $h(x) = f(x)g(x)$ is ***continuous*** at $x = c$.

   d) If $g(c) \neq 0$, then the function $h(x) = \dfrac{f(x)}{g(x)}$ is ***continuous*** at $x = c$.

2. For any polynomial $P(x)$, $\lim_{x \to c} P(x) = P(c)$, so ***all polynomials*** are ***continuous*** functions.

3. Given a polynomial $Q(x)$ with $Q(c) \neq 0$, then the $R(x) = \dfrac{P(x)}{Q(x)}$ is a ***continuous*** at $x = c$.

4. If the function $g(x)$ is ***continuous*** at $x = c$, and the function $f(x)$ is ***continuous*** at $x = g(c)$, then the function $h(x) = f(g(x))$ is ***continuous*** at $x = c$.