An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume II: Kinetics

Unit 1

Inertia Matrices (Dyadics), Angular Momentum and Kinetic Energy

Summary

This unit defines ***moments*** and ***products of inertia*** for rigid bodies and shows how they are used to form ***inertia matrices*** (or ***dyadics***). Inertia matrices are then used to calculate ***principal moments*** of ***inertia*** and ***principal directions***. More generally, it shows how to ***transform*** the ***components*** of ***inertia dyadics*** from ***one set*** of reference axes to ***another***. Finally, it defines ***angular momentum vectors*** and the ***kinetic energy function*** for rigid bodies and shows ***how*** to use inertia matrices to compute them.

An ***Addendum*** is included to discuss the special case of ***nondistinct*** (equal) ***principal moments*** of ***inertia*** and their ***associated eigenvectors***. The principal moments of inertia and the principal directions of a square prism are presented as an example.

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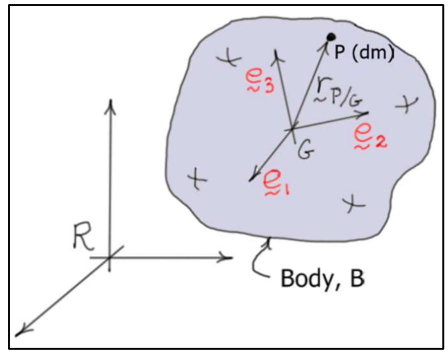
Moments and Products of Inertia and the Inertia Matrix

Moments of Inertia

Consider the rigid body B is shown in the diagram. The unit vectors $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ are fixed in B and are directed along a ***convenient*** set of axes (x,y,z) that pass through the mass center G. The ***moments of inertia*** of the body about these axes are defined as follows.



$$I_{xx}^G = \int_B (y^2+z^2)\,dm \qquad I_{yy}^G = \int_B (x^2+z^2)\,dm \qquad I_{zz}^G = \int_B (x^2+y^2)\,dm$$

Here, x, y, and z are defined as the $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, and $\underset{\sim}{e}_3$ components of $\underset{\sim}{r}_{P/G}$ the position vector of an arbitrary point P of the body relative to G, that is, $\underset{\sim}{r}_{P/G} = x\,\underset{\sim}{e}_1 + y\,\underset{\sim}{e}_2 + z\,\underset{\sim}{e}_3$. The integrals are taken over the ***entire volume*** of the mass.

Moments of inertia of a body about an axis measure the ***distribution*** of the body's ***mass about that axis*** and are always ***positive*** (although, if small (negligible), they can be ***assumed*** to be ***zero***). The ***smaller*** the inertia the more ***concentrated*** the mass is about the axis. Inertia values can be found by ***measurement***, ***calculation***, or ***both***. Calculations are based on ***direct integration***, the "***body build-up***" technique, or ***both***. In the body build-up technique, ***inertias*** of ***simple shapes*** are ***added*** together to estimate the inertia of a ***composite shape***. The inertias of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are ***transferred*** to axes through the composite mass center using the ***Parallel-Axes Theorem for Moments of Inertia***.

Parallel-Axes Theorem for Moments of Inertia

The inertia I_{ii}^A of a body about an axis i passing through any point A is equal to the ***sum*** of the inertia I_{ii}^G of the body about a parallel axis through the mass center G ***plus*** the mass m times the square of the ***shortest*** distance d_i between the two parallel axes.

$$I_{ii}^A = I_{ii}^G + m\,d_i^2 \quad (i = x, y, \text{ or } z)$$

As noted above, moments of inertia are ***always positive***. It is obvious from the parallel-axes theorem that the ***minimum moments of inertia*** of a body occur about axes passing through its ***mass center***. All other inertias must be larger as indicated by the addition of the term "$m\,d^2$".

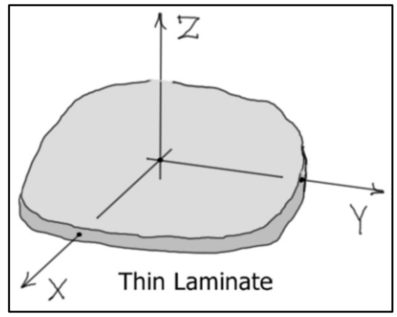
Products of Inertia

The ***products of inertia*** of the rigid body are measured relative to a ***pair*** of axes and are defined as follows

$$I_{xy}^{G} = I_{yx}^{G} = \int_B (xy)\, dm \qquad I_{xz}^{G} = I_{zx}^{G} = \int_B (xz)\, dm \qquad I_{yz}^{G} = I_{zy}^{G} = \int_B (yz)\, dm$$

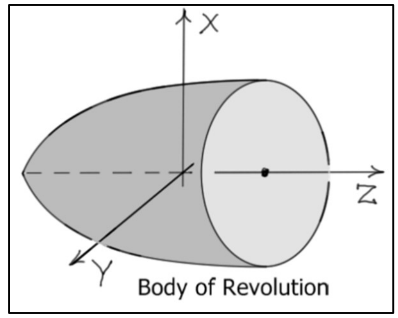
Again, the ***integrals*** are taken over the ***entire volume*** of the mass.

Products of inertia of a body are indicators of ***symmetry***. ***If a plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero***. For example, consider the ***thin laminate*** shown. The middle plane of the laminate lies in the *XY* plane so that half its thickness is above the plane and half is below. Hence, the *XY* plane is a ***plane of symmetry*** and



$$I_{xz} = I_{yz} = 0 \quad \text{(for a thin laminate)}$$

Bodies of revolution have ***two planes*** of ***symmetry***. For the configuration shown, the *XZ* and *YZ* planes are planes of symmetry. Hence, ***all products of inertia are zero*** about the *XYZ* axis system.



Products of inertia are found by ***measurement***, ***calculation***, or ***both***. Calculations are based on ***direct integration***, the "***body build-up***" technique, or ***both***. In the body build-up technique, ***products of inertia of simple shapes*** are ***added*** to estimate the products of inertia of a ***composite shape***. The products of inertia of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are transferred to axes through the composite mass center using the ***Parallel-Axes Theorem for Products of Inertia***.

Parallel-Axes Theorem for Products of Inertia

The product of inertia I_{ij}^{A} of a body about a pair of axes (i, j) passing through any point A is equal to the sum of the product of inertia I_{ij}^{G} of the body about a set of parallel axes through the mass center *G* plus the mass m times the product of the coordinates c_i and c_j of *G* relative to *A* (or *A* relative to *G*) measured along those axes.

$$I_{ij}^{A} = I_{ij}^{G} + m c_i c_j \qquad (i = x, y, \text{ or } z \text{ and } j = x, y, \text{ or } z)$$

Products of inertia can be ***positive***, ***negative***, or ***zero***.

The Inertia Matrix

The ***moments*** and ***products of inertia*** of a body about a set of axes (passing through some point) can be ***collected*** into a single ***inertia matrix***. For example, the inertia matrix of a body about a set of axes passing through its mass center G is defined as

$$[I_G] = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix}$$

Note that the ***diagonal entries*** are the ***moments of inertia*** and the ***off-diagonal entries*** are the ***negatives*** of the ***products of inertia***. Defining the matrix in this way is ***convenient*** for calculating the ***angular momentum*** of the body as discussed below.

For ***nonsymmetric bodies***, there can be an ***infinite number*** of inertia matrices associated with ***each point*** of a body because the inertia matrix changes with the orientation of the axes at that point. However, there is generally ***only one set*** of axes for each point for which the inertia matrix is ***diagonal***. These axes are called ***principal axes*** (or ***principal directions***) and the inertias about those axes are called ***principal moments*** of ***inertia*** for that point. In general, the ***principal axes*** and ***principal moments*** of ***inertia*** are ***different*** for ***each point*** of a body.

For ***symmetric bodies***, however, there can be ***multiple sets*** of ***principal axes*** at a given point and ***multiple points*** can have the ***same principal moments*** of ***inertia*** and ***principal axes***. For example, consider the body of revolution shown in the diagram above. As shown, the X and Y axes are principal axes for ***any point*** along the Z axis, and they can be rotated by any angle about the Z axis to produce another set of principal axes. Of course, any axis passing through the center of a sphere is a principal axis for that point.

All inertia matrices are ***symmetric***. Consequently, they have ***real eigenvalues*** and ***eigenvectors***. The ***eigenvalues*** of an inertia matrix are the ***principal moments*** of ***inertia*** and the ***eigenvectors*** are the ***principal directions*** for that point. If the eigenvectors are ***normalized,*** they represent a set of ***three mutually perpendicular unit vectors*** in the ***principal directions***.

The ***principal moments of inertia*** of a body for some point, say mass-center G, can be calculated by setting

$$\det\begin{bmatrix} (I_{xx}^G - \lambda) & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & (I_{yy}^G - \lambda) & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & (I_{zz}^G - \lambda) \end{bmatrix} = 0$$

By ***expanding*** the ***determinant***, the resulting ***characteristic equation*** can be written as follows.

$$\lambda^3 + \left(-I_{xx}^G - I_{yy}^G - I_{zz}^G\right)\lambda^2 + \left(I_{xx}^G I_{yy}^G + I_{xx}^G I_{zz}^G + I_{yy}^G I_{zz}^G - \left(I_{xy}^G\right)^2 - \left(I_{xz}^G\right)^2 - \left(I_{yz}^G\right)^2\right)\lambda$$
$$+\left(-I_{xx}^G I_{yy}^G I_{zz}^G + I_{xx}^G\left(I_{yz}^G\right)^2 + I_{yy}^G\left(I_{xz}^G\right)^2 + I_{zz}^G\left(I_{xy}^G\right)^2 + 2 I_{xy}^G I_{xz}^G I_{yz}^G\right) = 0$$

The ***three roots*** of this equation are the ***three principal moments of inertia***.

If $I_i^G \; (i=1,2,3)$ represent the three principal moments of inertia, the ***principal direction*** for each principal moment can be found by writing the following.

$$\begin{bmatrix} (I_{xx}^G - I_i^G) & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & (I_{yy}^G - I_i^G) & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & (I_{zz}^G - I_i^G) \end{bmatrix} \begin{Bmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{Bmatrix} = \{0\} \quad (i=1,2,3)$$

Since the coefficient matrix is ***singular***, these equations ***do not have a single solution***. The ***directions*** of the ***eigenvectors*** are ***unique***, but their ***magnitudes*** are ***not***. If the eigenvectors are taken to be ***unit vectors***, then $\boxed{a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1}$ and the vectors are unique. To solve for the components of each eigenvector, simply choose a value for one of the components, and then solve for the other two. Finally, ***normalize*** the resulting vector. The components of these normalized eigenvectors are the direction cosines for the principal directions.

It should be noted here that the process detailed above produces a ***unique set*** of ***three mutually perpendicular unit eigenvectors*** if all the ***eigenvalues*** are ***distinct*** (i.e., not equal). As noted above, for ***symmetric bodies*** that have ***principal inertia values*** that are ***not distinct*** (i.e. equal), the ***eigenvectors*** are ***not unique***. However, a set of mutually perpendicular unit eigenvectors can always be found. More details on this topic can be found in the Addendum to this Unit and in reference [3] (R.L. Huston, *Multibody Dynamics*, Butterworth-Heinemann, 1990).

Inertia matrices are also ***diagonalizable*** using their eigenvector (or modal) matrices. The columns of an eigenvector matrix $[M]$ associated with an inertia matrix $[I]$ are formed using the components of the normalized eigenvectors of $[I]$. The diagonal matrix of eigenvalues can then be calculated as follows.

$$[D] = [M]^T [I][M]$$

The eigenvalue associated with an eigenvector appears in the same column of $[D]$ as the eigenvector appears in $[M]$.

<u>The Inertia Dyadic</u>

The inertias of a body about a set of axes (passing through some point) can also be collected into a single ***inertia dyadic***. For example, the ***inertia dyadic*** of a body about a set of axes through its mass center G is defined as

$$\boxed{\underset{\approx}{I}_G = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \, \underset{\sim}{e}_i \, \underset{\sim}{e}_j}$$

Here, $\underset{\sim}{e}_i \; (i=1,2,3)$ are the ***unit vectors*** directed along the ***three axes***, and the ***components*** of the ***dyadic*** are the ***nine elements*** of the ***inertia matrix*** $I_{ij}^{G} \;\; (i,j=1,2,3)$. The vector products $\underset{\sim}{e}_i \, \underset{\sim}{e}_j \; (i,j=1,2,3)$ are called ***dyads***. This definition makes it clear that each inertia value is associated with a ***pair of axes***. For moments of inertia they are ***repeated pairs*** ((x,x), (y,y), or (z,z)), and for products of inertia they are ***non-repeated pairs*** ((x,y), (x,z), or (y,z)).

Properties of Dyads

Dyads satisfy many properties. Three very useful properties are

1. $\underset{\sim}{a}\,\underset{\sim}{b} \neq \underset{\sim}{b}\,\underset{\sim}{a}$
2. $\underset{\sim}{c}\cdot\left(\underset{\sim}{a}\,\underset{\sim}{b}\right)=\left(\underset{\sim}{c}\cdot\underset{\sim}{a}\right)\underset{\sim}{b}$ and $\left(\underset{\sim}{a}\,\underset{\sim}{b}\right)\cdot\underset{\sim}{c}=\underset{\sim}{a}\left(\underset{\sim}{b}\cdot\underset{\sim}{c}\right)=\left(\underset{\sim}{b}\cdot\underset{\sim}{c}\right)\underset{\sim}{a}$
3. $\left(\underset{\sim}{a}\,\underset{\sim}{b}+\underset{\sim}{c}\,\underset{\sim}{d}\right)\cdot\underset{\sim}{e}=\left(\underset{\sim}{b}\cdot\underset{\sim}{e}\right)\underset{\sim}{a}+\left(\underset{\sim}{d}\cdot\underset{\sim}{e}\right)\underset{\sim}{c}$

The latter two properties indicate that the "dot" product of a ***dyad*** and a ***vector*** is a ***vector***. Recall that the "dot" product of ***two vectors*** is a ***scalar***. These properties will be used later in the calculation of ***angular momentum*** of a body. The ***dyad-vector dot product*** is akin to the ***matrix-vector product*** of ***matrix algebra***.

Relationship between Dyadic Components in Different Frames

Like vectors, ***dyadics*** can be ***represented*** by ***components*** in ***different reference frames***. Consider the dyadic $\underset{\approx}{A}$ and its representations in two different reference frames $B:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ and $C:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$.

$$\boxed{\underset{\approx}{A}=\sum_{k,\ell=1}^{3} a_{k\ell}^{B}\,\underset{\sim}{n}_k\,\underset{\sim}{n}_\ell=\sum_{i,j=1}^{3} a_{ij}^{C}\,\underset{\sim}{e}_i\,\underset{\sim}{e}_j}$$

Here, $a_{k\ell}^{B}\;\left(k,\ell=1,2,3\right)$ represent the ***nine components*** of $\underset{\approx}{A}$ in $B:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$, and $a_{ij}^{C}\;\left(i,j=1,2,3\right)$ represent the ***nine components*** of $\underset{\approx}{A}$ in $C:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$. These ***two sets*** of components can be related by using the ***transformation matrix*** that relates the two reference frames.

If $\left[R\right]$ is the matrix that transforms vectors and their components ***from*** frame *C* ***into*** frame *B*, then

$$\sum_{i,j} a_{ij}^{C}\,\underset{\sim}{e}_i\,\underset{\sim}{e}_j=\sum_{i,j} a_{ij}^{C}\left(\sum_{k} R_{ik}^{T}\,\underset{\sim}{n}_k\right)\left(\sum_{\ell} R_{j\ell}^{T}\,\underset{\sim}{n}_\ell\right)=\sum_{k,\ell}\left(\sum_{i,j} a_{ij}^{C} R_{ik}^{T} R_{j\ell}^{T}\right)\underset{\sim}{n}_k\,\underset{\sim}{n}_\ell=\sum_{k,\ell} a_{k\ell}^{B}\,\underset{\sim}{n}_k\,\underset{\sim}{n}_\ell$$

Comparing the last two terms in this equation gives

$$\boxed{a_{k\ell}^{B}=\sum_{i,j} a_{ij}^{C} R_{ik}^{T} R_{j\ell}^{T}=\sum_{i,j} R_{ki}\, a_{ij}^{C}\, R_{j\ell}^{T}}$$

Note that the ***sums*** on indices i, j, k, and ℓ are all from 1 to 3, and the ***superscript*** *T* indicates the matrix transpose. The above result can be written in ***matrix form*** as

$$\boxed{\left[A^{B}\right]=\left[R\right]\left[A^{C}\right]\left[R\right]^{T}}$$

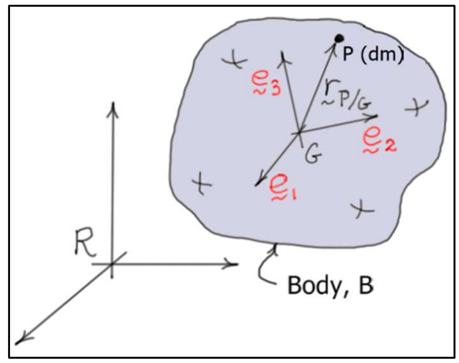
This result can be applied to the ***inertia matrix*** of rigid bodies. Given $\left[I_G^C\right]$ the inertia matrix of the body about a set of axes passing through G and parallel to $C:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$, $\left[I_G^B\right]$ the inertia matrix of a body about a second set of axes passing through its mass-center G and parallel to $B:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ can be calculated as follows

$$\boxed{\left[I_G^B\right]=[R]\left[I_G^C\right][R]^T}$$

As before, $[R]$ transforms vectors and their components from frame C into B.

Angular Momentum of a Rigid Body about its Mass Center

To calculate the ***angular momentum*** of a rigid body about its mass center G, consider the rigid body B. Point P represents ***an arbitrary point*** within the body, "*dm*" represents the elemental mass of the body associated with P, and $\underset{\sim}{r}_{P/G}$ represents the position vector of P with respect to G. The ***angular momentum*** of B about G is then defined as



$$\boxed{\underset{\sim}{H}_G=\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_P\right)dm}$$

The ***integral*** is taken over the ***entire volume*** of the mass.

An alternative form for $\underset{\sim}{H}_G$ can be found by using the kinematic formula for ***two points fixed*** on a ***rigid body*** and the ***definition of center of mass*** (i.e., $\int_B \underset{\sim}{r}_{P/G}\,dm=0$) as follows.

$$\underset{\sim}{H}_G=\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_P\right)dm=\int_B\left(\underset{\sim}{r}_{P/G}\times\left({}^R\underset{\sim}{v}_G+{}^R\underset{\sim}{v}_{P/G}\right)\right)dm=\left(\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_G\right)dm\right)+\left(\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_{P/G}\right)dm\right)$$

$$=\underbrace{\left(\int_B\underset{\sim}{r}_{P/G}\,dm\right)}_{\text{zero}}\times{}^R\underset{\sim}{v}_G+\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_{P/G}\right)dm$$

$$=\int_B\left(\underset{\sim}{r}_{P/G}\times\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{r}_{P/G}\right)\right)dm$$

$$\Rightarrow\boxed{\underset{\sim}{H}_G=\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_P\right)dm=\int_B\left(\underset{\sim}{r}_{P/G}\times{}^R\underset{\sim}{v}_{P/G}\right)dm=\int_B\left(\underset{\sim}{r}_{P/G}\times\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{r}_{P/G}\right)\right)dm}$$

This alternative form shows that the angular momentum $\underset{\sim}{H}_G$ incorporates ***only*** the ***angular motion*** of the body.

A ***more useful result*** that specifically relates $\underset{\sim}{H}_G$ to the concepts of ***inertia*** and ***angular velocity*** can be found by letting $\boxed{\underset{\sim}{r}_{P/G} = x\,\underset{\sim}{e}_1 + y\,\underset{\sim}{e}_2 + z\,\underset{\sim}{e}_3}$ and $\boxed{{}^R\underset{\sim}{\omega}_B = \omega_1\,\underset{\sim}{e}_1 + \omega_2\,\underset{\sim}{e}_2 + \omega_3\,\underset{\sim}{e}_3}$, and then using the ***vector identity*** $\boxed{\underset{\sim}{a}\times(\underset{\sim}{b}\times\underset{\sim}{c}) = (\underset{\sim}{a}\cdot\underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{c}}$ to expand the expression for $\underset{\sim}{H}_G$. In particular,

$$\begin{aligned}
\underset{\sim}{H}_G &= \int_B \left(\underset{\sim}{r}_{P/G}\times\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{r}_{P/G}\right)\right)dm = \int_B \left(\underset{\sim}{r}_{P/G}\cdot\underset{\sim}{r}_{P/G}\right){}^R\underset{\sim}{\omega}_B\,dm - \int_B \left(\underset{\sim}{r}_{P/G}\cdot{}^R\underset{\sim}{\omega}_B\right)\underset{\sim}{r}_{P/G}\,dm \\
&= \int_B \left(x^2+y^2+z^2\right)\left(\omega_1\,\underset{\sim}{e}_1+\omega_2\,\underset{\sim}{e}_2+\omega_3\,\underset{\sim}{e}_3\right)dm - \int_B \left(x\,\omega_1+y\,\omega_2+z\,\omega_3\right)\left(x\,\underset{\sim}{e}_1+y\,\underset{\sim}{e}_2+z\,\underset{\sim}{e}_3\right)dm \\
&= \int_B r^2\left(\omega_1\,\underset{\sim}{e}_1+\omega_2\,\underset{\sim}{e}_2+\omega_3\,\underset{\sim}{e}_3\right)dm - \int_B \left(x\omega_1+y\omega_2+z\omega_3\right)\left(x\,\underset{\sim}{e}_1+y\,\underset{\sim}{e}_2+z\,\underset{\sim}{e}_3\right)dm
\end{aligned}$$

Sorting the ***vector components*** gives

$$\begin{aligned}
\underset{\sim}{H}_G &= \int_B \left(r^2\omega_1 - x\left(x\omega_1+y\omega_2+z\omega_3\right)\right)\underset{\sim}{e}_1\,dm + \int_B \left(r^2\omega_2 - y\left(x\omega_1+y\omega_2+z\omega_3\right)\right)\underset{\sim}{e}_2\,dm + \\
&\qquad \int_B \left(r^2\omega_3 - z\left(x\,\omega_1+y\,\omega_2+z\,\omega_3\right)\right)\underset{\sim}{e}_3\,dm \\
&= \int_B \left(\left(y^2+z^2\right)\omega_1 - x\,y\,\omega_2 - x\,z\,\omega_3\right)\underset{\sim}{e}_1\,dm + \int_B \left(-x\,y\,\omega_1 + \left(x^2+z^2\right)\omega_2 - y\,z\,\omega_3\right)\underset{\sim}{e}_2\,dm + \\
&\qquad \int_B \left(-x\,z\,\omega_1 - y\,z\,\omega_2 + \left(x^2+y^2\right)\omega_3\right)\underset{\sim}{e}_3\,dm
\end{aligned}$$

The ***evaluation*** of the integrals ***does not depend*** on the ***angular velocity*** components or the ***unit vectors***, so the above equation can be further simplified as follows.

$$\begin{aligned}
\underset{\sim}{H}_G &= \left(\omega_1\int_B\left(y^2+z^2\right)dm + \omega_2\int_B\left(-x\,y\right)dm + \omega_3\int_B\left(-x\,z\right)dm\right)\underset{\sim}{e}_1 + \\
&\qquad \left(\omega_1\int_B\left(-x\,y\right)dm + \omega_2\int_B\left(x^2+z^2\right)dm + \omega_3\int_B\left(-y\,z\right)dm\right)\underset{\sim}{e}_2 + \\
&\qquad\qquad \left(\omega_1\int_B\left(-x\,z\right)dm + \omega_2\int_B\left(-y\,z\right)dm + \omega_3\int_B\left(x^2+y^2\right)dm\right)\underset{\sim}{e}_3
\end{aligned}$$

or

$$\boxed{\underset{\sim}{H}_G = \left(I_{xx}^G\,\omega_1 - I_{xy}^G\,\omega_2 - I_{xz}^G\,\omega_3\right)\underset{\sim}{e}_1 + \left(-I_{xy}^G\,\omega_1 + I_{yy}^G\,\omega_2 - I_{yz}^G\,\omega_3\right)\underset{\sim}{e}_2 + \left(-I_{xz}^G\,\omega_1 - I_{yz}^G\,\omega_2 + I_{zz}^G\,\omega_3\right)\underset{\sim}{e}_3}$$

Here the ***integrals*** are now recognized as the ***moments*** and ***products of inertia*** of the body about axes parallel to $\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ and passing through the mass center *G*.

Note that for ***two-dimensional motion*** the ***angular momentum*** of a body is in the ***same direction*** as the ***angular velocity*** of the body, both being ***normal*** to the ***plane of motion***. In ***three-dimensional motion***, however, the ***angular momentum*** is generally ***not*** in the ***same direction*** as the ***angular velocity***. This contrasts with the ***linear momentum*** of a body which is in the ***same direction*** as the velocity of the mass center of the body for both two and three-dimensional motion.

Representation of Angular Momentum as a Matrix-Vector Product

The above result for the angular momentum vector $\underset{\sim}{H}_G$ is easier to remember when we note the following ***matrix-vector product*** can be used to generate the components.

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{e}_1 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{e}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{e}_3 \end{Bmatrix} = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

Here, the inertias and angular velocity components must be ***resolved*** (calculated) about the ***same*** set of ***directions*** in this case indicated by the unit vectors $\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$.

Representation of Angular Momentum as a Dyadic-Vector Product

The angular momentum vector $\underset{\sim}{H}_G$ can also be written as the "dot" product of the ***inertia dyadic*** with the ***angular velocity*** vector. That is,

$$\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B$$

This is easily verified by substituting for $\underset{\approx}{I}_G$ and ${}^R\underset{\sim}{\omega}_B$ in this expression and expanding.

$$\underset{\sim}{H}_G = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \, \underset{\sim}{e}_i \, \underset{\sim}{e}_j \right) \cdot \left(\sum_{k=1}^{3} \omega_k \, \underset{\sim}{e}_k \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \omega_k \, I_{ij}^G \, \underset{\sim}{e}_i \left(\underset{\sim}{e}_j \cdot \underset{\sim}{e}_k \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \omega_k \, I_{ij}^G \, \underset{\sim}{e}_i \delta_{jk}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \, \omega_j \, \underset{\sim}{e}_i$$

$$= \left(I_{11}^G \omega_1 + I_{12}^G \omega_2 + I_{13}^G \omega_3 \right) \underset{\sim}{e}_1 + \left(I_{21}^G \omega_1 + I_{22}^G \omega_2 + I_{23}^G \omega_3 \right) \underset{\sim}{e}_2 + \left(I_{31}^G \omega_1 + I_{32}^G \omega_2 + I_{33}^G \omega_3 \right) \underset{\sim}{e}_3$$

$$= \left(I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3 \right) \underset{\sim}{e}_1 + \left(-I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3 \right) \underset{\sim}{e}_2 + \left(-I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3 \right) \underset{\sim}{e}_3$$

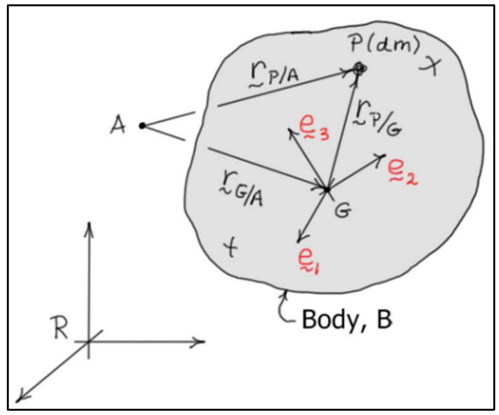
Here, δ_{jk} (often called Kronecker's delta function) is equal to ***one*** when $j = k$ and ***zero*** when $j \neq k$.

This result is the same as that obtained using the matrix-vector product presented above. However, note that the ***unit vectors*** used in the analysis appear ***explicitly*** in the ***dyadic-vector product***, whereas they ***do not*** in the ***matrix-vector product***. Obviously either approach produces correct results, but ***care*** must be taken when using the ***matrix-vector product*** to ensure the ***same directions*** are used for both the inertia matrix and angular velocity components. The resulting ***angular momentum components*** are in the directions of the ***same set of unit vectors***.

Angular Momentum of a Rigid Body about an Arbitrary Point

The angular momentum of a rigid body about an ***arbitrary point*** A is defined as

$$\boxed{\underset{\sim}{H}_A = \int_B \left(\underset{\sim}{r}_{P/A} \times {}^R\underset{\sim}{v}_P \right) dm}$$



Here, $\underset{\sim}{r}_{P/A}$ is the position vector of ***points*** P within the body relative to A, and again, the ***integral*** is taken over the ***entire volume*** of the mass. The angular momentum $\underset{\sim}{H}_A$ can be ***related to*** the angular momentum $\underset{\sim}{H}_G$ by recognizing that $\boxed{\underset{\sim}{r}_{P/A} = \underset{\sim}{r}_{G/A} + \underset{\sim}{r}_{P/G}}$.

Substituting this expression into the integral and expanding gives

$$\begin{aligned} \underset{\sim}{H}_A &= \int_B \left(\left(\underset{\sim}{r}_{G/A} + \underset{\sim}{r}_{P/G} \right) \times {}^R\underset{\sim}{v}_P \right) dm = \int_B \left(\underset{\sim}{r}_{G/A} \times {}^R\underset{\sim}{v}_P \right) dm + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P \right) dm \\ &= \underset{\sim}{r}_{G/A} \times \left(\int_B {}^R\underset{\sim}{v}_P \, dm \right) + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P \right) dm \\ &= \underset{\sim}{r}_{G/A} \times \left(m\, {}^R\underset{\sim}{v}_G \right) \quad + \quad \underset{\sim}{H}_G \end{aligned}$$

or

$$\boxed{\underset{\sim}{H}_A = \underset{\sim}{H}_G \ + \ \underset{\sim}{r}_{G/A} \times m\, {}^R\underset{\sim}{v}_G}$$

The last term in this expression represents the ***moment*** of the ***linear momentum*** of the body about A (assuming the line of action of the linear momentum vector passes through G).

<u>Special Case: Motion about a Fixed Point on the Body</u>

If some point O of the rigid body is ***fixed*** so the body ***pivots*** about that point, then the velocity of the mass center can be written as ${}^R\underset{\sim}{v}_G = \underbrace{{}^R\underset{\sim}{v}_O}_{\text{zero}} + {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O}$. Substituting this result into the definition for angular momentum gives

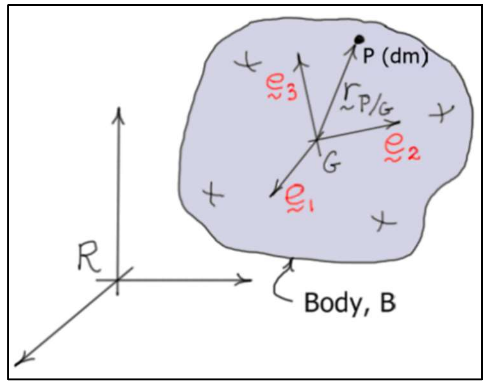
$$\boxed{\underset{\sim}{H}_O = \int_B \left(\underset{\sim}{r}_{P/O} \times {}^R\underset{\sim}{v}_P \right) dm = \int_B \left(\underset{\sim}{r}_{P/O} \times {}^R\underset{\sim}{v}_{P/O} \right) dm = \int_B \left(\underset{\sim}{r}_{P/O} \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/O} \right) \right) dm}$$

This expression is ***like*** that obtained for the ***mass center*** except the position vector is ***referenced*** to the ***fixed-point*** O. Hence, the angular momentum about O is computed in the same way as for the mass center except the ***inertia values*** are ***measured*** about O. That is,

$$\boxed{\underset{\sim}{H}_O = \underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\omega}_B}$$

Here, $\underset{\approx}{I}_O$ is the inertia dyadic (or matrix) about the fixed-point O. If ***moments*** and ***products of inertia*** for axes passing through the mass center are ***known***, then the ***parallel-axes theorems*** for moments and products of inertia can be used to compute $\underset{\approx}{I}_O$.

Kinetic Energy of a Rigid Body



The figure at the right depicts a rigid body B moving relative to a fixed frame R. The ***kinetic energy*** of B is defined as

$$\boxed{K = \int_B \tfrac{1}{2}\left({}^R\underset{\sim}{v}_P \cdot {}^R\underset{\sim}{v}_P\right) dm}$$

Here, ${}^R\underset{\sim}{v}_P$ is the velocity of ***points*** P of the body, and the ***integral*** is taken over the ***entire volume*** of the mass.

A more ***useful definition*** can be derived by ***relating*** the velocity of P to the velocity of the mass center G. Using the ***relative velocity equation***, the integrand can be rewritten as

$$\begin{aligned}{}^R\underset{\sim}{v}_P \cdot {}^R\underset{\sim}{v}_P &= \left({}^R\underset{\sim}{v}_P\right)^2 = \left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{P/G}\right)^2 = \left({}^R\underset{\sim}{v}_G + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{r}_{P/G}\right)\right)^2 \\ &= \left({}^R\underset{\sim}{v}_G\right)^2 + 2\,{}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{r}_{P/G}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{r}_{P/G}\right)^2\end{aligned}$$

Substituting back into the integral gives the following three terms:

1. $\displaystyle\int_B \tfrac{1}{2}\left({}^R\underset{\sim}{v}_G\right)^2 dm = \tfrac{1}{2}\left({}^R\underset{\sim}{v}_G\right)^2 \underbrace{\int_B dm}_{m} = \tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G\right)^2 = \tfrac{1}{2} m\, v_G^2$

2. $\displaystyle\int_B 2\,{}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right) dm = 2\,{}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_B \times \underbrace{\left(\int_B \underset{\sim}{r}_{P/G}\, dm\right)}_{\text{zero}}\right) = 0$ … (definition of mass center)

3. Letting $\underset{\sim}{r}_{P/G} = x\,\underset{\sim}{e}_1 + y\,\underset{\sim}{e}_2 + z\,\underset{\sim}{e}_3$ and ${}^R\underset{\sim}{\omega}_B = \omega_1\,\underset{\sim}{e}_1 + \omega_2\,\underset{\sim}{e}_2 + \omega_3\,\underset{\sim}{e}_3$, the integrand of the third integral can be expanded as follows:

$$\begin{aligned}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)^2 &= \left(\omega_2 z - \omega_3 y\right)^2 + \left(\omega_3 x - \omega_1 z\right)^2 + \left(\omega_1 y - \omega_2 x\right)^2 \\ &= \omega_1^2\left(y^2+z^2\right) + \omega_2^2\left(x^2+z^2\right) + \omega_3^2\left(x^2+y^2\right) - 2\,\omega_1\,\omega_2\,x\,y - 2\,\omega_1\,\omega_3\,x\,z - 2\,\omega_2\,\omega_3\,y\,z\end{aligned}$$

Substituting into the integral gives

$$\begin{aligned}\int_B \tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)^2 dm &= \tfrac{1}{2}\omega_1^2\int_B\left(y^2+z^2\right)dm + \tfrac{1}{2}\omega_2^2\int_B\left(x^2+z^2\right)dm + \tfrac{1}{2}\omega_3^2\int_B\left(x^2+y^2\right)dm \\ &\quad - \omega_1\,\omega_2\int_B x\,y\,dm - \omega_1\,\omega_3\int_B x\,z\,dm - \omega_2\,\omega_3\int_B y\,z\,dm \\ &= \tfrac{1}{2}\omega_1^2 I_{xx}^G + \tfrac{1}{2}\omega_2^2 I_{yy}^G + \tfrac{1}{2}\omega_3^2 I_{zz}^G - \omega_1\,\omega_2 I_{xy}^G - \omega_1\,\omega_3 I_{xz}^G - \omega_2\,\omega_3 I_{yz}^G\end{aligned}$$

It is ***easy to show*** that this last result is equal to $\frac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G$.

Adding the three terms gives the following result for the kinetic energy.

$$K = \underbrace{\tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G\right)^2}_{\text{translational energy}} + \underbrace{\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G}_{\text{rotational energy}} = \tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G\right)^2 + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B$$

Special Case: Motion about a Fixed-Point *O*

If there is a point O within the body that is ***fixed*** so that the body pivots about O, then

$$\left({}^R\underset{\sim}{v}_G\right)^2 = \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O}\right)^2 = \left(y_G^2 + z_G^2\right)\omega_1^2 + \left(x_G^2 + z_G^2\right)\omega_2^2 + \left(x_G^2 + y_G^2\right)\omega_3^2$$
$$- 2\,\omega_1\,\omega_2\,x_G\,y_G - 2\,\omega_1\,\omega_3\,x_G\,z_G - 2\,\omega_2\,\omega_3\,y_G\,z_G$$

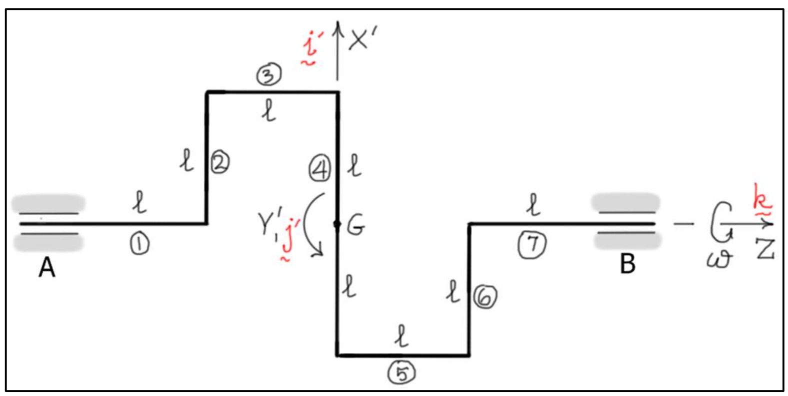
Substituting this result into the boxed equation above and combining terms, the kinetic energy can be reduced to ***purely rotational energy*** about O.

$$K = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_O = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\omega}_B$$

Here $\underset{\approx}{I}_O$ is the inertia dyadic (or matrix) for a set of axes passing through the fixed-point O.

Example 1: Angular Momentum and Kinetic Energy of a Simple Crank Shaft

The figure shows a ***simple crank shaft*** consisting of ***seven segments***, each considered to be a ***slender bar***. Each segment of ***length*** ℓ has ***mass*** m. There are six segments of length ℓ and one segment of length 2ℓ (segment 4). The mass center of the system G is located on the axis of rotation.



Reference frames:

R: $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ (fixed frame)

S: $\underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k}$ (rotates with the shaft)

Find:

a) $\underset{\sim}{H}_G$ the angular momentum of the system about its mass center, G

b) K the kinetic energy of the system

Solution:

a) The elements of the ***inertia matrix*** can be found using the ***parallel-axes theorems*** for moments and products of inertia and the ***body build-up technique***. However, given that the angular velocity of the system is only about the Z axis, only the ***third column*** of the inertia matrix need be determined. Specifically, the S frame components of $\underset{\sim}{H}_G$ can be written as follows.

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{i}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{j}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{k} \end{Bmatrix} = \begin{bmatrix} I^G_{X'X'} & -I^G_{X'Y'} & -I^G_{X'Z} \\ -I^G_{Y'X'} & I^G_{Y'Y'} & -I^G_{Y'Z} \\ -I^G_{ZX'} & -I^G_{ZY'} & I^G_{ZZ} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} = \begin{Bmatrix} -I^G_{X'Z}\omega \\ -I^G_{Y'Z}\omega \\ I^G_{ZZ}\omega \end{Bmatrix}$$

Using the ***parallel-axes theorem*** for ***moments of inertia*** and the body build-up technique, the moment of inertia of the system about the Z axis can be calculated as follows

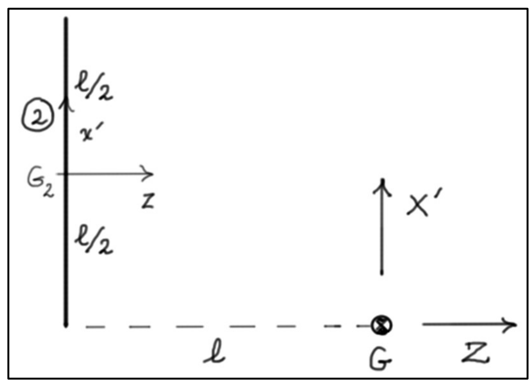
$$I_{ZZ}^{G} = \sum_{i=1}^{7} \left(I_{ZZ}^{G} \right)_i = 0 + \tfrac{1}{3} m\ell^2 + m\ell^2 + \tfrac{1}{12}(2m)(2\ell)^2 + m\ell^2 + \tfrac{1}{3} m\ell^2 + 0 \quad \Rightarrow \boxed{I_{ZZ}^{G} = \tfrac{10}{3} m\ell^2}$$

The contributions of each of the ***seven segments*** are shown individually in the equation. Segments 1 and 7 lie along the Z-axis and as slender bars have ***approximately zero*** inertia about that axis. The Z-axis passes along the ends of segments 2 and 6 so they each contribute $\frac{1}{3} m\ell^2$. Segments 3 and 5 are parallel to the Z-axis at a distance of ℓ so they each contribute approximately $m\ell^2$, and the Z-axis passes through the mass center of segment 4 so it contributes $\frac{1}{12}(2m)(2\ell)^2$.

Since the $X'Z$ plane is a ***plane of symmetry***, the products of inertia associated with the Y' direction are zero. Hence, $I_{Y'Z}^{G} \equiv 0$. The product $I_{X'Z}^{G}$, however, is ***not zero***. It can be calculated using the ***parallel-axes theorem*** for ***products of inertia*** and the body build-up technique as follows

$$I_{X'Z}^{G} = \sum_{i=1}^{7} \left(I_{X'Z}^{G} \right)_i = 0 + m(\tfrac{\ell}{2})(-\ell) + m(\ell)(-\tfrac{\ell}{2}) + 0 + m(-\ell)(\tfrac{\ell}{2}) + m(-\tfrac{\ell}{2})(\ell) + 0 \quad \Rightarrow \boxed{I_{X'Z}^{G} = -2m\ell^2}$$

Again, the contributions of each of the ***seven links*** are shown individually in the equation. To calculate the contribution of each link, imagine a set of ***local axes*** passing through the mass centers of each of the segments and parallel to the X' and Z axes. (The products of inertia of each of the segments about their ***local axes*** are ***zero*** due to ***symmetry***.) Then apply the parallel-axes theorem to find the products of inertia about the system's mass center. For example, the product of inertia of segment 2 about the system's mass center G can be calculated as follows



$$\boxed{\left(I_{X'Z}^{G} \right)_2 = \underbrace{\left(I_{x'z}^{G_2} \right)_2}_{\text{zero}} + m\left(\tfrac{\ell}{2}\right)(-\ell) = -\tfrac{1}{2} m\ell^2}$$

The first term is the product of inertia of segment 2 about its mass center axes (which, again, is zero due to symmetry) and the second term is the product of "m" times the product of the X' and Z coordinates of G_2 relative to G. Note that the product of the X' and Z coordinates of G relative to G_2 produces the same result. A similar approach is taken with each of the segments.

Substituting these results into the expression for $\underset{\sim}{H}_G$ gives

$$\boxed{\underset{\sim}{H}_G = 2m\ell^2\omega\, \underset{\sim}{i}' + (\tfrac{10}{3}) m\ell^2\omega\, \underset{\sim}{k}}$$

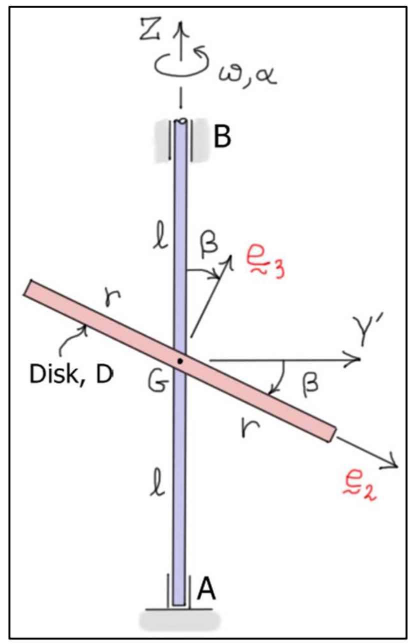
Note here that even though the ***angular motion*** is only about the ***Z axis***, the ***angular momentum*** has a component which is ***normal to*** that direction due to the ***mass asymmetry*** of the system about the directions of the *S*-frame. Mass asymmetries such as this induce ***oscillatory loads*** on the support bearings. At significant rotational speeds, these loads cause the supporting structure to ***vibrate***. The support loads for this system are calculated in Unit 2 of this volume.

b) The ***kinetic energy*** of the crank shaft is found from the ***velocity*** and ***angular momentum*** vectors to be

$$K = \underbrace{\tfrac{1}{2}m({}^R\underset{\sim}{v}_G)^2}_{\text{zero}} + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2}(\omega\underset{\sim}{k})\cdot \underset{\sim}{H}_G = \tfrac{10}{6}m\ell^2\omega^2 \quad \Rightarrow \boxed{K = \tfrac{10}{6}m\ell^2\omega^2}$$

From its definition, it is clear the ***kinetic energy*** of a body incorporates only the ***component*** of the angular momentum which is in the ***direction*** of the ***angular velocity***.

Example 2: Angular Momentum and Kinetic Energy of a Misaligned Disk (or Gear)



The system shown consists of two bodies, shaft *AB* of length 2ℓ and disk *D* of radius r. *D* is ***welded*** to *AB* so that an axis normal to *D* makes an angle β with the shaft axis. A non-zero angle β indicates the disk is not aligned properly on the shaft. The shaft and disk rotate together about the *Z* axis at a rate of ω (r/s). The mass center of the disk is on the axis of rotation.

Reference frames: (*R* is the fixed frame)

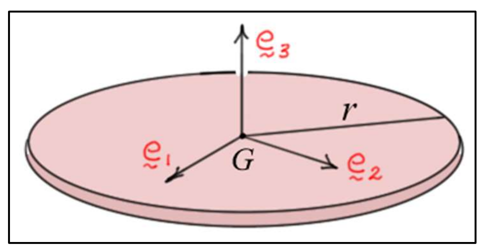
$S:\ \underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k}$ (rotates with the shaft; aligned with the shaft)

$D:\ \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ (rotates with the shaft; aligned with the disk) $\left(\underset{\sim}{e}_1 = \underset{\sim}{i}'\right)$

Find:

a) $\underset{\sim}{H}_G$ the ***angular momentum*** of the disk about its mass center, *G*
b) K the ***kinetic energy*** of the disk

Solution:



a) Note the reference frame $D:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ represents a set of ***principal axes*** for the disk. Assuming the disk is ***thin***, its inertia matrix relative to the *D*-frame axes can be written as

$$\boxed{\left[I_G\right]_D = m r^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}$$ (all products of inertia are zero due to symmetry)

Here a subscript *D* has been used to indicate the ***reference directions*** are ***fixed*** in the ***disk***. To calculate the angular momentum using this result, the components of the angular velocity vector must be resolved in *D* as well. Using S_β and C_β to represent the $\sin(\beta)$ and $\cos(\beta)$, the angular velocity can be written as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_D = \omega\,\underset{\sim}{k} = \omega\left(-S_\beta\,\underset{\sim}{e}_2 + C_\beta\,\underset{\sim}{e}_3\right)}$$

Substituting into the definition of angular momentum gives the components of $\underset{\sim}{H}_G$ resolved in D.

$$\begin{Bmatrix} \underset{\sim}{H}_G\cdot\underset{\sim}{e}_1 \\ \underset{\sim}{H}_G\cdot\underset{\sim}{e}_2 \\ \underset{\sim}{H}_G\cdot\underset{\sim}{e}_3 \end{Bmatrix} = m\,r^2\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} 0 \\ -\omega S_\beta \\ \omega C_\beta \end{Bmatrix} = m\,r^2\begin{Bmatrix} 0 \\ -\frac{1}{4}\omega S_\beta \\ \frac{1}{2}\omega C_\beta \end{Bmatrix} \quad\Rightarrow \boxed{\underset{\sim}{H}_G = \tfrac{1}{4}m\,r^2\omega\left(-S_\beta\,\underset{\sim}{e}_2 + 2C_\beta\,\underset{\sim}{e}_3\right)}$$

The shaft-based components of $\underset{\sim}{H}_G$ can now be found by recognizing from the diagram that $\underset{\sim}{e}_2 = C_\beta\,\underset{\sim}{j}' - S_\beta\,\underset{\sim}{k}$ and $\underset{\sim}{e}_3 = S_\beta\,\underset{\sim}{j}' + C_\beta\,\underset{\sim}{k}$. Substituting these into the above expression gives

$$\underset{\sim}{H}_G = \tfrac{1}{4}m\,r^2\omega\left(-S_\beta\,\underset{\sim}{e}_2 + 2C_\beta\,\underset{\sim}{e}_3\right) = \tfrac{1}{4}m\,r^2\omega\left[-S_\beta\left(C_\beta\,\underset{\sim}{j}' - S_\beta\,\underset{\sim}{k}\right) + 2C_\beta\left(S_\beta\,\underset{\sim}{j}' + C_\beta\,\underset{\sim}{k}\right)\right]$$

or

$$\boxed{\underset{\sim}{H}_G = \tfrac{1}{4}m\,r^2\omega\left[\left(S_\beta C_\beta\right)\underset{\sim}{j}' + \left(2C_\beta^2 + S_\beta^2\right)\underset{\sim}{k}\right] = \tfrac{1}{4}m\,r^2\omega\left[\left(S_\beta C_\beta\right)\underset{\sim}{j}' + \left(C_\beta^2 + 1\right)\underset{\sim}{k}\right]}$$

As with the system of Example 1, the ***angular momentum*** has a component which is ***normal to*** the angular velocity due to the ***mass asymmetry*** of the system about the directions of the *S*-frame. If the misalignment angle is set to zero, $\underset{\sim}{H}_G$ reverts to a simpler form which is in the direction of the angular velocity.

$$\underset{\sim}{H}_G = \tfrac{1}{4}m\,r^2\omega\left[\left(S_\beta C_\beta\right)\underset{\sim}{j}' + \left(C_\beta^2 + 1\right)\underset{\sim}{k}\right]_{\beta=0} = \tfrac{1}{4}m\,r^2\omega\left[2\,\underset{\sim}{k}\right] \quad\Rightarrow \boxed{\underset{\sim}{H}_G = \tfrac{1}{2}m\,r^2\omega\,\underset{\sim}{k}}$$

Note here that the ***angular momentum*** was calculated about the ***shaft-based system*** without first finding the inertias about those axes. However, the above result can be used to determine these inertias by noting that

$$\boxed{\underset{\sim}{H}_G = \left(-I_{X'Z}\omega\right)\underset{\sim}{i}' + \left(-I_{Y'Z}\omega\right)\underset{\sim}{j}' + \left(I_{ZZ}\omega\right)\underset{\sim}{k} = \tfrac{1}{4}m\,r^2\omega\left[\left(S_\beta C_\beta\right)\underset{\sim}{j}' + \left(C_\beta^2 + 1\right)\underset{\sim}{k}\right]}$$

Equating each of the ***vector components*** leads to the following inertias about the shaft axes.

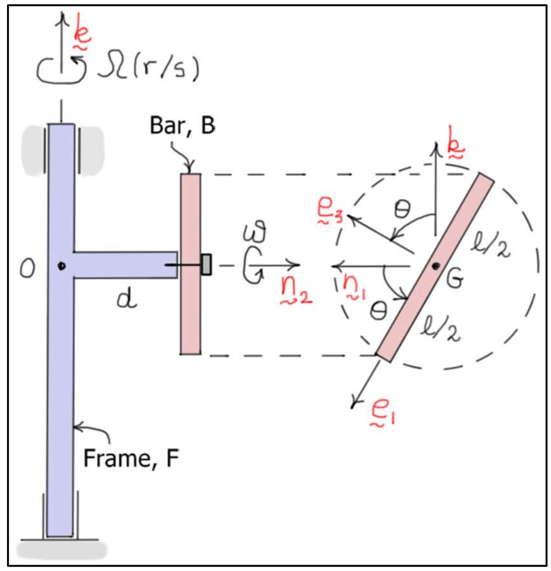
$$\boxed{I_{X'Z} = 0},\ \boxed{I_{Y'Z} = -\tfrac{1}{4}m\,r^2 S_\beta C_\beta},\ \boxed{I_{ZZ} = \tfrac{1}{4}m\,r^2\left(C_\beta^2 + 1\right)}$$

b) The ***kinetic energy*** of the disk is found from the ***velocity*** and ***angular momentum*** vectors to be

$$K = \underbrace{\tfrac{1}{2}m({}^R\underset{\sim}{v}_G)^2}_{\text{zero}} + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_D\cdot\underset{\sim}{H}_G = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_D\cdot\underset{\sim}{H}_G = \tfrac{1}{2}(\omega\underset{\sim}{k})\cdot\underset{\sim}{H}_G = \tfrac{1}{8}mr^2\left(C_\beta^2 + 1\right)\omega^2 \quad\Rightarrow \boxed{K = \tfrac{1}{8}mr^2\left(C_\beta^2 + 1\right)\omega^2}$$

Again, note that the ***kinetic energy*** involves only the ***component*** of the angular momentum in the ***direction*** of the ***angular velocity***.

Example 3: Angular Momentum and Kinetic Energy of a Rotating Bar



The system shown consists of ***two bodies***, frame *F* and bar *B*. Frame *F* rotates about the ***fixed vertical direction*** annotated by the unit vector $\underset{\sim}{k}$. Bar *B* is pinned to and rotates about the horizontal arm of *F*. *F* rotates relative to the ground at a rate $\Omega\ (\text{r/s})$ and *B* rotates relative to *F* at a rate of $\omega\ (\text{r/s})$.

Reference frames: (*R* is the fixed frame)

$F: \underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{k}$ (rotates with frame *F*)

$B: \underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{e}_3$ (rotates with the bar *B*)

Find:

a) $\underset{\sim}{H}_G$ the ***angular momentum*** of *B* about its mass center, *G*

b) K the ***kinetic energy*** of *B*

Solution:

Assuming the bar is ***slender***, the inertia matrix of the bar about its mass center *G* associated with frame $B:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{e}_3)$ can be written as follows. Note again that the subscript *B* indicates the reference directions are fixed in *B*.

$$\boxed{[I_G]_B = \tfrac{1}{12} m\,\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

To use this result to find $\underset{\sim}{H}_G$, the angular velocity must also be resolved into ***body-fixed components***. Using the ***summation rule*** for angular velocities and S_θ and C_θ to represent the $\sin(\theta)$ and $\cos(\theta)$, the angular velocity can be written as

$$ {}^R\underset{\sim}{\omega}_B = {}^F\underset{\sim}{\omega}_D + {}^R\underset{\sim}{\omega}_F = \omega\,\underset{\sim}{n}_2 + \Omega\,\underset{\sim}{k} = \omega\,\underset{\sim}{n}_2 + \Omega(-S_\theta \underset{\sim}{e}_1 + C_\theta \underset{\sim}{e}_3) \quad \Rightarrow \boxed{{}^R\underset{\sim}{\omega}_B = (-\Omega S_\theta)\underset{\sim}{e}_1 + \omega\,\underset{\sim}{n}_2 + (\Omega C_\theta)\underset{\sim}{e}_3} $$

The ***body-fixed components*** of the ***angular momentum*** vector can then be calculated as follows.

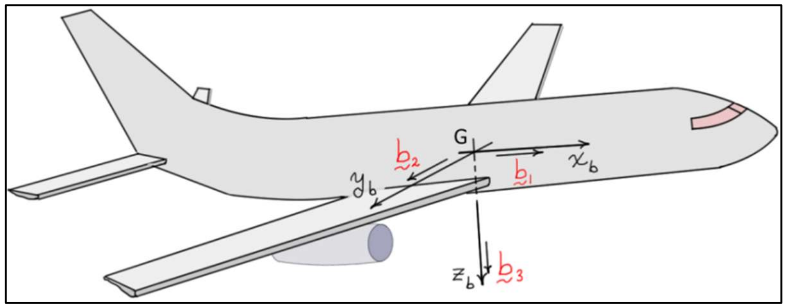
$$ \begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{e}_1 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{n}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{e}_3 \end{Bmatrix} = \frac{m\ell^2}{12} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -\Omega S_\theta \\ \omega \\ \Omega C_\theta \end{Bmatrix} = \frac{m\ell^2}{12} \begin{Bmatrix} 0 \\ \omega \\ \Omega C_\theta \end{Bmatrix} \quad \Rightarrow \boxed{\underset{\sim}{H}_G = \frac{m\ell^2}{12}\left(\omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right)} $$

b) The ***kinetic energy*** of the bar includes both ***translational*** and ***rotational*** energy. Using the angular velocity and angular momentum vectors and noting that ${}^R\underset{\sim}{v}_G = -d\,\Omega\,\underset{\sim}{n}_1$, the kinetic energy can now be written as

$$ K = \tfrac{1}{2} m ({}^R\underset{\sim}{v}_G)^2 + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2} m d^2 \Omega^2 + \tfrac{1}{2}\left(-\Omega S_\theta\,\underset{\sim}{e}_1 + \omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right) \cdot \tfrac{m\ell^2}{12}\left(\omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right) $$

$$ \Rightarrow \boxed{K = \tfrac{1}{2} m d^2 \Omega^2 + \tfrac{m\ell^2}{24}\left(\omega^2 + \Omega^2 C_\theta^2\right)} $$

Example 4: Aircraft with Two Engines



The aircraft shown has two engines, one on each wing. The orientation of the aircraft relative to a fixed reference frame R is defined by a 3-2-1 ***body-fixed*** rotation sequence (ψ, θ, ϕ). For the purposes of this example, the ***aircraft*** is made up of ***three*** main components, the ***airframe*** A and the ***two engines*** E_1 and E_2. The term ***airframe*** is used to refer to all the ***stationary components*** of the aircraft, and the term ***engine*** is used to refer to the ***rotating components*** of the engines. The points $G_i \ (i=1,2)$ are the mass centers of the two ***engines***, G_A is the mass center of the ***airframe***, and G is the mass center of the ***aircraft***.

The aircraft is ***symmetrical*** with respect to the $x_b z_b$ plane. The ***two engines*** are assumed to be ***identical*** and ***placed symmetrically*** on the airframe so the position vectors of $G_i \ (i=1,2)$ relative to G the mass center of the aircraft can be written as $\underset{\sim}{r}_{G_1/G} = x_E \underset{\sim}{b}_1 + y_E \underset{\sim}{b}_2 + z_E \underset{\sim}{b}_3$ and $\underset{\sim}{r}_{G_2/G} = x_E \underset{\sim}{b}_1 - y_E \underset{\sim}{b}_2 + z_E \underset{\sim}{b}_3$. The engines (rotating components) are assumed to be ***solids of revolution*** aligned with the x_b axis (meaning they are ***rotationally symmetrical*** about that axis). Finally, the ***velocity*** of the mass center of the aircraft is given in ***body-frame*** as ${}^R\underset{\sim}{v}_G = u \underset{\sim}{b}_1 + v \underset{\sim}{b}_2 + w \underset{\sim}{b}_3$ and in the ***ground-frame*** as ${}^R\underset{\sim}{v}_G = \dot{X} \underset{\sim}{N}_1 + \dot{Y} \underset{\sim}{N}_2 + \dot{Z} \underset{\sim}{N}_3$.

Reference frames:

$R: \underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3$ (inertial or ground frame)

$A: \underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$ (frame fixed in the aircraft)

Find: (express vector components in frame A)

a) $\underset{\sim}{H}_{G_A}$ the angular momentum of the airframe about its mass center G_A
b) $\underset{\sim}{H}_{G_i} \ (i=1,2)$ the angular momenta of the engines about their mass centers $G_i \ (i=1,2)$
c) $\underset{\sim}{H}_G$ the angular momentum of the aircraft (airframe and engines) about its mass center G
d) K the kinetic energy of the aircraft

Solution:

a) Assuming the $x_b z_b$ plane is a ***plane of symmetry*** of the airframe, its inertia matrix about its mass center G_A can be written as

$$\left[I_{G_A}\right]_A = \begin{bmatrix} I_{x_b x_b}^{G_A} & -I_{x_b y_b}^{G_A} & -I_{x_b z_b}^{G_A} \\ -I_{y_b x_b}^{G_A} & I_{y_b y_b}^{G_A} & -I_{y_b z_b}^{G_A} \\ -I_{z_b x_b}^{G_A} & -I_{z_b y_b}^{G_A} & I_{z_b z_b}^{G_A} \end{bmatrix} = \begin{bmatrix} I_{x_b x_b}^{G_A} & 0 & -I_{x_b z_b}^{G_A} \\ 0 & I_{y_b y_b}^{G_A} & 0 \\ -I_{x_b z_b}^{G_A} & 0 & I_{z_b z_b}^{G_A} \end{bmatrix}$$

Recall the subscript A on the inertia matrix indicates its elements are measured about ***airframe-fixed axes***, and note that because the aircraft is symmetrical about the $x_b z_b$ plane, $I_{x_b y_b}^{G_A} = I_{y_b z_b}^{G_A} = 0$. In Unit 5 of Volume I the ***angular velocity*** of a body whose orientation is described using a 3-2-1, body-fixed orientation angle sequence was found to be

$$\boxed{{}^R\underset{\sim}{\omega}_A = \omega_1 \underset{\sim}{b}_1 + \omega_2 \underset{\sim}{b}_2 + \omega_3 \underset{\sim}{b}_3 = \left(\dot{\phi} - \dot{\psi}\, S_\theta\right) \underset{\sim}{b}_1 + \left(\dot{\theta}\, C_\phi + \dot{\psi}\, C_\theta\, S_\phi\right) \underset{\sim}{b}_2 + \left(-\dot{\theta}\, S_\phi + \dot{\psi}\, C_\theta\, C_\phi\right) \underset{\sim}{b}_3}$$

The ***body-fixed components*** of the ***angular momentum*** of the airframe can then be calculated as follows

$$\begin{Bmatrix} \underset{\sim}{H}_{G_A} \cdot \underset{\sim}{b}_1 \\ \underset{\sim}{H}_{G_A} \cdot \underset{\sim}{b}_2 \\ \underset{\sim}{H}_{G_A} \cdot \underset{\sim}{b}_3 \end{Bmatrix} = \begin{bmatrix} I_{x_b x_b}^{G_A} & 0 & -I_{x_b z_b}^{G_A} \\ 0 & I_{y_b y_b}^{G_A} & 0 \\ -I_{x_b z_b}^{G_A} & 0 & I_{z_b z_b}^{G_A} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} I_{x_b x_b}^{G_A} \omega_1 - I_{x_b z_b}^{G_A} \omega_3 \\ I_{y_b y_b}^{G_A} \omega_2 \\ -I_{x_b z_b}^{G_A} \omega_1 + I_{z_b z_b}^{G_A} \omega_3 \end{Bmatrix}$$

$$\Rightarrow \boxed{\underset{\sim}{H}_{G_A} = \left(I_{x_b x_b}^{G_A} \omega_1 - I_{x_b z_b}^{G_A} \omega_3\right) \underset{\sim}{b}_1 + \left(I_{y_b y_b}^{G_A} \omega_2\right) \underset{\sim}{b}_2 + \left(-I_{x_b z_b}^{G_A} \omega_1 + I_{z_b z_b}^{G_A} \omega_3\right) \underset{\sim}{b}_3}$$

b) Given the ***rotating components*** of the engines are ***solids of revolution*** whose axes are parallel to $\underset{\sim}{b}_1$, the ***inertia matrices*** of the engines about a set of axes parallel to reference frame A and passing through the mass centers of the engines can be written as

$$\boxed{\left[I_{G_i}\right]_A = \begin{bmatrix} I_{x_b x_b}^{G_i} & -I_{x_b y_b}^{G_i} & -I_{x_b z_b}^{G_i} \\ -I_{y_b x_b}^{G_i} & I_{y_b y_b}^{G_i} & -I_{y_b z_b}^{G_i} \\ -I_{z_b x_b}^{G_i} & -I_{z_b y_b}^{G_i} & I_{z_b z_b}^{G_i} \end{bmatrix} = \begin{bmatrix} I_{x_b x_b}^{E} & 0 & 0 \\ 0 & I_{y_b y_b}^{E} & 0 \\ 0 & 0 & I_{z_b z_b}^{E} \end{bmatrix}} \quad (i = 1, 2)$$

Due to the assumed ***rotational symmetry*** of the engines about the $\underset{\sim}{b}_1$ direction, all products of inertia are zero, and the inertias $I_{y_b y_b}^{E}$ and $I_{z_b z_b}^{E}$ are ***equal*** and ***constant*** relative to directions fixed in the airframe A.

The ***angular velocities*** of the engines can be calculated using the ***summation rule*** for angular velocities.

$${}^R\underset{\sim}{\omega}_{E_i} = {}^R\underset{\sim}{\omega}_A + {}^A\underset{\sim}{\omega}_{E_i} = (\omega_1 \underset{\sim}{b}_1 + \omega_2 \underset{\sim}{b}_2 + \omega_3 \underset{\sim}{b}_3) + (\omega_{E_i} \underset{\sim}{b}_1)$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{\omega}_{E_i} = (\omega_1 + \omega_{E_i}) \underset{\sim}{b}_1 + \omega_2 \underset{\sim}{b}_2 + \omega_3 \underset{\sim}{b}_3} \quad (i = 1, 2)$$

Using the above results, the ***aircraft-fixed components*** of the ***angular momenta*** of the engines can then be calculated as follows

$$\begin{Bmatrix} \underset{\sim}{H}_{G_i} \cdot \underset{\sim}{b}_1 \\ \underset{\sim}{H}_{G_i} \cdot \underset{\sim}{b}_2 \\ \underset{\sim}{H}_{G_i} \cdot \underset{\sim}{b}_3 \end{Bmatrix} = \begin{bmatrix} I_{x_b x_b}^{E} & 0 & 0 \\ 0 & I_{y_b y_b}^{E} & 0 \\ 0 & 0 & I_{z_b z_b}^{E} \end{bmatrix} \begin{Bmatrix} \omega_1 + \omega_{E_i} \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} I_{x_b x_b}^{E} (\omega_1 + \omega_{E_i}) \\ I_{y_b y_b}^{E} \omega_2 \\ I_{z_b z_b}^{E} \omega_3 \end{Bmatrix}$$

$$\Rightarrow \boxed{\underset{\sim}{H}_{G_i} = \left(I_{x_b x_b}^{E} (\omega_1 + \omega_{E_i})\right) \underset{\sim}{b}_1 + \left(I_{y_b y_b}^{E} \omega_2\right) \underset{\sim}{b}_2 + \left(I_{z_b z_b}^{E} \omega_3\right) \underset{\sim}{b}_3} \quad (i = 1, 2)$$

c) The angular momentum of the aircraft about its mass center G is the ***sum*** of the angular momenta of the airframe and the two engines about G.

$$\boxed{\underset{\sim}{H}_G = \left(\underset{\sim}{H}_G\right)_A + \sum_{i=1}^{2}\left(\underset{\sim}{H}_G\right)_{E_i}}$$

The angular momentum of the ***airframe*** about G the mass center of the ***aircraft*** can be calculated as follows.

$$\begin{aligned}
\left(\underset{\sim}{H}_G\right)_A &= \underset{\sim}{H}_{G_A} + \left(\underset{\sim}{r}_{G_A/G} \times m_A\, {}^R\underset{\sim}{v}_{G_A}\right) = \underset{\sim}{H}_{G_A} + \left(\underset{\sim}{r}_{G_A/G} \times m_A\left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{G_A/G}\right)\right)\\
&= \underset{\sim}{H}_{G_A} + \left(\underset{\sim}{r}_{G_A/G} \times m_A\, {}^R\underset{\sim}{v}_{G_A/G}\right) + \left(\underset{\sim}{r}_{G_A/G} \times m_A\, {}^R\underset{\sim}{v}_G\right)\\
&= \underset{\sim}{H}_{G_A} + m_A\left(\underset{\sim}{r}_{G_A/G} \times \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)\right) + \left(m_A \underset{\sim}{r}_{G_A/G}\right) \times {}^R\underset{\sim}{v}_G
\end{aligned}$$

The ***second term*** on the right side can be ***expanded*** using the vector identity $\boxed{\underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right) = \left(\underset{\sim}{a} \cdot \underset{\sim}{c}\right)\underset{\sim}{b} - \left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)\underset{\sim}{c}}$ and letting $\underset{\sim}{r}_{G_A/G} = x_A \underset{\sim}{b}_1 + z_A \underset{\sim}{b}_3$ gives the following.

$$\begin{aligned}
m_A\left(\underset{\sim}{r}_{G_A/G} \times \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)\right) &= m_A\left(\left(\underset{\sim}{r}_{G_A/G} \cdot \underset{\sim}{r}_{G_A/G}\right){}^R\underset{\sim}{\omega}_A - \left(\underset{\sim}{r}_{G_A/G} \cdot {}^R\underset{\sim}{\omega}_A\right)\underset{\sim}{r}_{G_A/G}\right)\\
&= m_A\left(x_A^2 + z_A^2\right)\left(\omega_1 \underset{\sim}{b}_1 + \omega_2 \underset{\sim}{b}_2 + \omega_3 \underset{\sim}{b}_3\right) - \left(x_A \omega_1 + z_A \omega_3\right)\left(x_A \underset{\sim}{b}_1 + z_A \underset{\sim}{b}_3\right)\\
&= m_A\left[\left(x_A^2 + z_A^2\right)\omega_1 - \left(x_A^2\right)\omega_1 - \left(x_A z_A\right)\omega_3\right]\underset{\sim}{b}_1 + m_A\left[\left(x_A^2 + z_A^2\right)\omega_2\right]\underset{\sim}{b}_2\\
&\quad + m_A\left[\left(x_A^2 + z_A^2\right)\omega_3 - \left(x_A z_A\right)\omega_1 - \left(z_A^2\right)\omega_3\right]\underset{\sim}{b}_3
\end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} m_A\left(\underset{\sim}{r}_{G_A/G} \times \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)\right) &= m_A\left[\left(z_A^2\right)\omega_1 - \left(x_A z_A\right)\omega_3\right]\underset{\sim}{b}_1 + m_A\left[\left(x_A^2 + z_A^2\right)\omega_2\right]\underset{\sim}{b}_2\\ &\quad + m_A\left[-\left(x_A z_A\right)\omega_1 + \left(x_A^2\right)\omega_3\right]\underset{\sim}{b}_3\end{aligned}}$$

Note that the $\underset{\sim}{b}_2$ component of the position vector $\underset{\sim}{r}_{G_A/G}$ is ***zero*** because the engines are symmetrically placed with respect to the $x_b z_b$ plane.

Combining this term with $\underset{\sim}{H}_{G_A}$ and using the parallel-axes theorems for moments and products of inertia gives

$$\begin{aligned}
&\underset{\sim}{H}_{G_A} + m_A\left(\underset{\sim}{r}_{G_A/G} \times \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)\right)\\
&= \begin{bmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \end{bmatrix} \begin{bmatrix} I_{x_b x_b}^{G_A} + m_A z_A^2 & 0 & -\left(I_{x_b z_b}^{G_A} + m_A x_A z_A\right)\\ 0 & I_{y_b y_b}^{G_A} + m_A\left(x_A^2 + z_A^2\right) & 0\\ -\left(I_{x_b z_b}^{G_A} + m_A x_A z_A\right) & 0 & I_{z_b z_b}^{G_A} + m_A x_A^2 \end{bmatrix} \begin{Bmatrix} \omega_1\\ \omega_2\\ \omega_3 \end{Bmatrix}\\
&= \begin{bmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \end{bmatrix} \begin{bmatrix} \left(I_{x_b x_b}^{G}\right)_A & 0 & -\left(I_{x_b z_b}^{G}\right)_A\\ 0 & \left(I_{y_b y_b}^{G}\right)_A & 0\\ -\left(I_{x_b z_b}^{G}\right)_A & 0 & \left(I_{z_b z_b}^{G}\right)_A \end{bmatrix} \begin{Bmatrix} \omega_1\\ \omega_2\\ \omega_3 \end{Bmatrix}
\end{aligned}$$

$$\Rightarrow \boxed{\underset{\sim}{H}_{G_A} + m_A\left(\underset{\sim}{r}_{G_A/G} \times \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)\right) = \left(\underset{\approx}{I}_G\right)_A \cdot {}^R\underset{\sim}{\omega}_A}$$

Here, $\left(\underset{\approx}{I}_G\right)_A$ represents the ***inertia tensor*** of the ***airframe*** as measured about G the mass center of the aircraft.

Substituting this result into the original expression for $\left(\underset{\sim}{H}_G\right)_A$ gives

$$\boxed{\left(\underset{\sim}{H}_G\right)_A = \left[\left(\underset{\approx}{I}_G\right)_A \cdot {}^R\underset{\sim}{\omega}_A\right] + \left(m_A \underset{\sim}{r}_{G_A/G}\right) \times {}^R\underset{\sim}{v}_G}$$

Using a ***similar procedure*** for each of the engines along with the ***summation rule*** for angular velocities, the expressions for the ***angular momenta*** of the ***engines*** can be simplified as follows.

$$\begin{aligned}
\left(\underset{\sim}{H}_G\right)_{E_i} &= \underset{\sim}{H}_{G_i} + \left(\underset{\sim}{r}_{G_i/G} \times m_E\, {}^R\underset{\sim}{v}_{G_i}\right) = \left(\underset{\approx}{I}_{G_i} \cdot {}^R\underset{\sim}{\omega}_{E_i}\right) + \left(\underset{\sim}{r}_{G_i/G} \times m_E\left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{G_i/G}\right)\right) \\
&= \left(\underset{\approx}{I}_{G_i} \cdot \left({}^R\underset{\sim}{\omega}_A + {}^A\underset{\sim}{\omega}_{E_i}\right)\right) + \left(\underset{\sim}{r}_{G_i/G} \times m_E\, {}^R\underset{\sim}{v}_{G_i/G}\right) + \left(\underset{\sim}{r}_{G_i/G} \times m_E\, {}^R\underset{\sim}{v}_G\right) \\
&= \underbrace{\left(\underset{\approx}{I}_{G_i} \cdot {}^R\underset{\sim}{\omega}_A\right) + m_E\left(\underset{\sim}{r}_{G_i/G} \times \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_i/G}\right)\right)} + \left(\underset{\approx}{I}_{G_i} \cdot {}^A\underset{\sim}{\omega}_{E_i}\right) + \left(m_E \underset{\sim}{r}_{G_i/G}\right) \times {}^R\underset{\sim}{v}_G
\end{aligned}$$

$$\Rightarrow \boxed{\left(\underset{\sim}{H}_G\right)_{E_i} = \left[\left(\underset{\approx}{I}_G\right)_{E_i} \cdot {}^R\underset{\sim}{\omega}_A\right] + \left(\underset{\approx}{I}_{G_i} \cdot {}^A\underset{\sim}{\omega}_{E_i}\right) + \left(m_E \underset{\sim}{r}_{G_i/G}\right) \times {}^R\underset{\sim}{v}_G}$$

Substituting all terms into the equation for $\underset{\sim}{H}_G$ gives

$$\begin{aligned}
\underset{\sim}{H}_G &= \left(\underset{\sim}{H}_G\right)_A + \sum_{i=1}^{2}\left(\underset{\sim}{H}_G\right)_{E_i} \\
&= \left[\left(\underset{\approx}{I}_G\right)_A \cdot {}^R\underset{\sim}{\omega}_A\right] + \left[\left(m_A \underset{\sim}{r}_{G_A/G}\right) \times {}^R\underset{\sim}{v}_G\right] + \left[\left(\underset{\approx}{I}_G\right)_{E_1} \cdot {}^R\underset{\sim}{\omega}_A\right] + \left(\underset{\approx}{I}_{G_1} \cdot {}^A\underset{\sim}{\omega}_{E_1}\right) + \left[\left(m_E \underset{\sim}{r}_{G_1/G}\right) \times {}^R\underset{\sim}{v}_G\right] \\
&\quad + \left[\left(\underset{\approx}{I}_G\right)_{E_2} \cdot {}^R\underset{\sim}{\omega}_A\right] + \left(\underset{\approx}{I}_{G_2} \cdot {}^A\underset{\sim}{\omega}_{E_2}\right) + \left[\left(m_E \underset{\sim}{r}_{G_2/G}\right) \times {}^R\underset{\sim}{v}_G\right] \\
&= \left[\left(\underset{\approx}{I}_G\right)_A + \left(\underset{\approx}{I}_G\right)_{E_1} + \left(\underset{\approx}{I}_G\right)_{E_2}\right] \cdot {}^R\underset{\sim}{\omega}_A + \left(\underset{\approx}{I}_{G_1} \cdot {}^A\underset{\sim}{\omega}_{E_1}\right) + \left(\underset{\approx}{I}_{G_2} \cdot {}^A\underset{\sim}{\omega}_{E_2}\right) \\
&\quad + \left[\underbrace{\left(m_A \underset{\sim}{r}_{G_A/G} + m_E \underset{\sim}{r}_{G_1/G} + m_E \underset{\sim}{r}_{G_2/G}\right)}_{\text{zero}} \times {}^R\underset{\sim}{v}_G\right]
\end{aligned}$$

$$\Rightarrow \boxed{\underset{\sim}{H}_G = \left[\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\right] \cdot {}^R\underset{\sim}{\omega}_A + \left(\underset{\approx}{I}_{G_1} \cdot {}^A\underset{\sim}{\omega}_{E_1}\right) + \left(\underset{\approx}{I}_{G_2} \cdot {}^A\underset{\sim}{\omega}_{E_2}\right)}$$

Here, $\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}$ represents the inertia tensor of the entire aircraft about its mass center G, and using the definition of ***center of mass***, the sum $m_A \underset{\sim}{r}_{G_A/G} + m_E \underset{\sim}{r}_{G_1/G} + m_E \underset{\sim}{r}_{G_2/G}$ is recognized to be ***zero***. A more ***specific form*** for the ***airframe-fixed components*** of $\underset{\sim}{H}_G$ can be written as follows.

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{b}_1 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{b}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{b}_3 \end{Bmatrix} = \begin{bmatrix} I^G_{x_b x_b} & 0 & -I^G_{x_b z_b} \\ 0 & I^G_{y_b y_b} & 0 \\ -I^G_{x_b z_b} & 0 & I^G_{z_b z_b} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} + \begin{bmatrix} I^E_{x_b x_b} & 0 & 0 \\ 0 & I^E_{y_b y_b} & 0 \\ 0 & 0 & I^E_{z_b z_b} \end{bmatrix} \begin{Bmatrix} \omega_{E_1} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} I^E_{x_b x_b} & 0 & 0 \\ 0 & I^E_{y_b y_b} & 0 \\ 0 & 0 & I^E_{z_b z_b} \end{bmatrix} \begin{Bmatrix} \omega_{E_2} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{b}_1 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{b}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{b}_3 \end{Bmatrix} = \begin{Bmatrix} I^G_{x_b x_b}\omega_1 - I^G_{x_b z_b}\omega_3 + I^E_{x_b x_b}\omega_{E_1} + I^E_{x_b x_b}\omega_{E_2} \\ I^G_{y_b y_b}\omega_2 \\ -I^G_{x_b z_b}\omega_1 + I^G_{z_b z_b}\omega_3 \end{Bmatrix}}$$

Note here that $I^G_{ij}\ (i, j = x_b, y_b \text{ or } z_b)$ represent moments and products of inertia of the ***entire aircraft*** about its mass center G while $I^E_{x_b x_b}$ represents the moments of inertia of just the ***rotating components*** of the engines about their axes of rotation. As defined above, variables $\omega_{E_i}\ (i = 1, 2)$ represent the rotational speeds of the engines ***relative*** to the aircraft.

d) The ***kinetic energy*** of the aircraft is the ***sum*** of the kinetic energies of the airframe and its two engines.

$$\boxed{\begin{aligned} K &= K_A + \sum_{i=1}^{2} K_{E_i} \\ &= \left(\tfrac{1}{2} m_A \left({}^R\underset{\sim}{v}_{G_A}\right)^2 + \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_A \cdot \underset{\sim}{H}_{G_A}\right) + \sum_{i=1}^{2}\left(\tfrac{1}{2} m_E \left({}^R\underset{\sim}{v}_{G_i}\right)^2 + \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_{E_i} \cdot \underset{\sim}{H}_{G_i}\right) \end{aligned}}$$

The above expression has ***three translational kinetic energy terms*** and ***three rotational kinetic energy terms***. It can be transformed into an expression with a ***single*** translational energy term associated with G the mass center of the aircraft as follows. First, rewrite the translational energy of the airframe and the engines in terms of ${}^R\underset{\sim}{v}_G$.

$$\boxed{\begin{aligned} \tfrac{1}{2} m_A \left({}^R\underset{\sim}{v}_{G_A}\right)^2 &= \tfrac{1}{2} m_A \left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{G_A/G}\right)^2 = \tfrac{1}{2} m_A \left({}^R\underset{\sim}{v}_G + \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)\right)^2 \\ &= \tfrac{1}{2} m_A \left({}^R\underset{\sim}{v}_G\right)^2 + \tfrac{1}{2} m_A \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right)^2 + {}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_A \times m_A \underset{\sim}{r}_{G_A/G}\right) \end{aligned}}$$

Similarly, for the translational kinetic energies of the engines

$$\boxed{\begin{aligned} \tfrac{1}{2} m_E \left({}^R\underset{\sim}{v}_{G_i}\right)^2 &= \tfrac{1}{2} m_E \left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{G_i/G}\right)^2 = \tfrac{1}{2} m_E \left({}^R\underset{\sim}{v}_G + \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_i/G}\right)\right)^2 \\ &= \tfrac{1}{2} m_E \left({}^R\underset{\sim}{v}_G\right)^2 + \tfrac{1}{2} m_E \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_i/G}\right)^2 + {}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_A \times m_E \underset{\sim}{r}_{G_i/G}\right) \end{aligned}} \quad (i = 1, 2)$$

Summing these three terms gives

$$\tfrac{1}{2}m_A\left({}^R\underset{\sim}{v}_{G_A}\right)^2+\sum_{i=1}^{2}\tfrac{1}{2}m_E\left({}^R\underset{\sim}{v}_{G_i}\right)^2$$

$$=\tfrac{1}{2}m_A\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2+{}^R\underset{\sim}{v}_G\cdot\left({}^R\underset{\sim}{\omega}_A\times m_A\underset{\sim}{r}_{G_A/G}\right)$$

$$+\sum_{i=1}^{2}\left[\tfrac{1}{2}m_E\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)^2+{}^R\underset{\sim}{v}_G\cdot\left({}^R\underset{\sim}{\omega}_A\times m_E\underset{\sim}{r}_{G_i/G}\right)\right]$$

$$=\tfrac{1}{2}\left(m_A+2m_E\right)\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)^2+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)^2$$

$$+{}^R\underset{\sim}{v}_G\cdot\left[{}^R\underset{\sim}{\omega}_A\times\underbrace{\left(m_A\underset{\sim}{r}_{G_A/G}+m_E\underset{\sim}{r}_{G_1/G}+m_E\underset{\sim}{r}_{G_2/G}\right)}_{\text{zero}}\right]$$

$$\Rightarrow\boxed{\begin{aligned}\tfrac{1}{2}m_A\left({}^R\underset{\sim}{v}_{G_A}\right)^2+\sum_{i=1}^{2}\tfrac{1}{2}m_E\left({}^R\underset{\sim}{v}_{G_i}\right)^2&=\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2\\&\quad+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)^2+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)^2\end{aligned}}$$

Here, $m_T=m_A+2m_E$ is the ***total mass*** of the ***aircraft*** and, using the definition of center of mass, the sum in square brackets is recognized to be ***zero***. Finally, substituting this result into the original expression for K gives

$$\boxed{\begin{aligned}K=\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2&+\left[\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{H}_{G_A}+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2\right]\\&+\left[\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_{E_1}\cdot\underset{\sim}{H}_{G_1}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)^2\right]+\left[\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_{E_2}\cdot\underset{\sim}{H}_{G_2}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)^2\right]\end{aligned}}$$

The three terms in square brackets on the right side of this result can be ***further simplified*** as follows. Consider the first bracketed term associated with the airframe and recall the ***vector identity*** $\boxed{\left(\underset{\sim}{a}\times\underset{\sim}{b}\right)\cdot\underset{\sim}{c}=\underset{\sim}{a}\cdot\left(\underset{\sim}{b}\times\underset{\sim}{c}\right)}$.

$$\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{H}_{G_A}+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{H}_{G_A}+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\cdot\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)$$

$$=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{H}_{G_A}+\tfrac{1}{2}m_A\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\sim}{r}_{G_A/G}\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\right)$$

$$=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left[\underset{\sim}{H}_{G_A}+m_A\left(\underset{\sim}{r}_{G_A/G}\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\right)\right]$$

Using the results found above for the term in square brackets gives

$$\boxed{\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{H}_{G_A}+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_A\cdot{}^R\underset{\sim}{\omega}_A}$$

Following a ***similar process*** for the engines along with the ***summation rule*** for angular velocities gives

$$\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_{E_i}\cdot\underset{\sim}{H}_{G_i}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{r}_{G_i/G}\right)^2=\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A+{}^A\underset{\sim}{\omega}_{E_i}\right)\cdot\underset{\sim}{H}_{G_i}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)\cdot\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)$$

$$=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_{G_i}\cdot\left({}^R\underset{\sim}{\omega}_A+{}^A\underset{\sim}{\omega}_{E_i}\right)\right)+\tfrac{1}{2}\,{}^A\underset{\sim}{\omega}_{E_i}\cdot\left(\underset{\approx}{I}_{G_i}\cdot\left({}^R\underset{\sim}{\omega}_A+{}^A\underset{\sim}{\omega}_{E_i}\right)\right)+\tfrac{1}{2}m_E\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\sim}{r}_{G_i/G}\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)\right)$$

$$=\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}m_E\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\sim}{r}_{G_i/G}\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)\right)+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_i}\cdot{}^A\underset{\sim}{\omega}_{E_i}\right)$$
$$+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_i}\cdot\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_i}\cdot\underset{\approx}{I}_{G_i}\cdot{}^A\underset{\sim}{\omega}_{E_i}\right)$$

$$=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\omega}_A\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)\right)\right]+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_i}\cdot{}^A\underset{\sim}{\omega}_{E_i}\right)$$
$$+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_i}\cdot\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_i}\cdot\underset{\approx}{I}_{G_i}\cdot{}^A\underset{\sim}{\omega}_{E_i}\right)$$

Again, using results found above, write

$$\Rightarrow\boxed{\begin{aligned}&\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_{E_i}\cdot\underset{\sim}{H}_{G_i}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_i/G}\right)^2\\&\quad=\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_{E_i}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_i}\cdot{}^A\underset{\sim}{\omega}_{E_i}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_i}\cdot\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_i}\cdot\underset{\approx}{I}_{G_i}\cdot{}^A\underset{\sim}{\omega}_{E_i}\right)\end{aligned}}$$

Substituting into the kinetic energy function and simplifying gives

$$K=\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2+\left[\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{H}_{G_A}+\tfrac{1}{2}m_A\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)^2\right]$$
$$+\left[\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_{E_1}\cdot\underset{\sim}{H}_{G_1}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)^2\right]+\left[\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_{E_2}\cdot\underset{\sim}{H}_{G_2}+\tfrac{1}{2}m_E\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)^2\right]$$
$$=\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}\left[{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_A\cdot{}^R\underset{\sim}{\omega}_A\right]$$
$$+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_{E_1}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}\right)$$
$$+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_{E_2}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}\right)$$
$$=\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}\left[{}^R\underset{\sim}{\omega}_A\cdot\left(\left(\underset{\approx}{I}_G\right)_A+\left(\underset{\approx}{I}_G\right)_{E_1}+\left(\underset{\approx}{I}_G\right)_{E_2}\right)\cdot{}^R\underset{\sim}{\omega}_A\right]$$
$$+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}\right)$$
$$+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}\right)$$

$$\Rightarrow\boxed{\begin{aligned}K=&\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2+\tfrac{1}{2}\left[{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot{}^R\underset{\sim}{\omega}_A\right]\\&+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}\right)\\&+\tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\omega}_A\right)+\tfrac{1}{2}\left({}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}\right)\end{aligned}}$$

This ***general expression*** above can be reduced to a ***more specific result*** as follows. Considering each term individually:

$$\boxed{\tfrac{1}{2}m_T\left({}^R\underset{\sim}{v}_G\right)^2=\tfrac{1}{2}m_T\left(u^2+v^2+w^2\right)=\tfrac{1}{2}m_T\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)}$$

$$\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}=\begin{bmatrix} I^G_{x_bx_b} & 0 & -I^G_{x_bz_b} \\ 0 & I^G_{y_by_b} & 0 \\ -I^G_{x_bz_b} & 0 & I^G_{z_bz_b} \end{bmatrix}$$

$$\begin{aligned}\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot{}^R\underset{\sim}{\omega}_A&=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left[\left(I^G_{x_bx_b}\omega_1-I^G_{x_bz_b}\omega_3\right)\underset{\sim}{b}_1+\left(I^G_{y_by_b}\omega_2\right)\underset{\sim}{b}_2+\left(-I^G_{x_bz_b}\omega_1+I^G_{z_bz_b}\omega_3\right)\underset{\sim}{b}_3\right]\\&=\tfrac{1}{2}\left[\left(I^G_{x_bx_b}\omega_1-I^G_{x_bz_b}\omega_3\right)\omega_1+\left(I^G_{y_by_b}\omega_2\right)\omega_2+\left(-I^G_{x_bz_b}\omega_1+I^G_{z_bz_b}\omega_3\right)\omega_3\right]\end{aligned}$$

$$\Rightarrow\boxed{\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot{}^R\underset{\sim}{\omega}_A=\tfrac{1}{2}\left[I^G_{x_bx_b}\omega_1^2+I^G_{y_by_b}\omega_2^2+I^G_{z_bz_b}\omega_3^2-2I^G_{x_bz_b}\omega_1\,\omega_3\right]}$$

$$\boxed{\tfrac{1}{2}\,{}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}=\tfrac{1}{2}\,\omega_{E_1}\underset{\sim}{n}_1\cdot\underset{\approx}{I}_{G_1}\cdot\omega_{E_1}\underset{\sim}{n}_1=\tfrac{1}{2}I^E_{x_bx_b}\omega^2_{E_1}}$$

$$\boxed{\tfrac{1}{2}\,{}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}=\tfrac{1}{2}\,\omega_{E_2}\underset{\sim}{n}_1\cdot\underset{\approx}{I}_{G_2}\cdot\omega_{E_2}\underset{\sim}{n}_1=\tfrac{1}{2}I^E_{x_bx_b}\omega^2_{E_2}}$$

$$\boxed{\tfrac{1}{2}\,{}^A\underset{\sim}{\omega}_{E_1}\cdot\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\omega}_A=\tfrac{1}{2}I^E_{x_bx_b}\omega_{E_1}\omega_1=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_1}\cdot{}^A\underset{\sim}{\omega}_{E_1}}$$

$$\boxed{\tfrac{1}{2}\,{}^A\underset{\sim}{\omega}_{E_2}\cdot\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\omega}_A=\tfrac{1}{2}I^E_{x_bx_b}\omega_{E_2}\omega_1=\tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_A\cdot\underset{\approx}{I}_{G_2}\cdot{}^A\underset{\sim}{\omega}_{E_2}}$$

Substituting into the general expression for K gives the final detailed result.

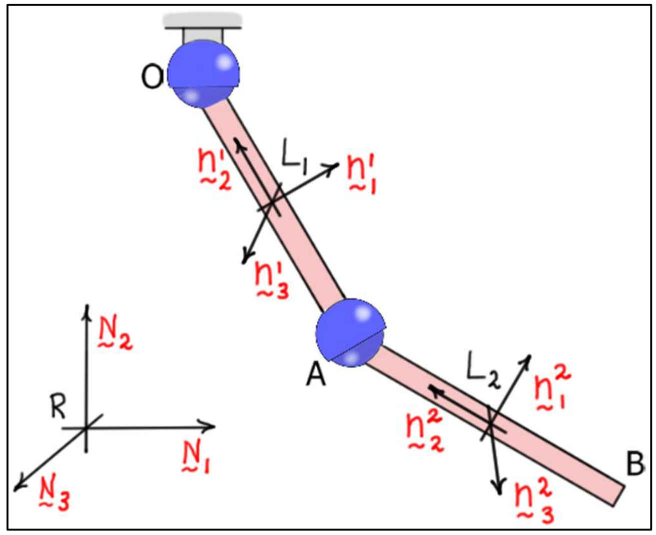
$$\boxed{\begin{aligned}K=\tfrac{1}{2}m_T\left(u^2+v^2+w^2\right)&+\tfrac{1}{2}\left[I^G_{x_bx_b}\omega_1^2+I^G_{y_by_b}\omega_2^2+I^G_{z_bz_b}\omega_3^2-2I^G_{x_bz_b}\omega_1\,\omega_3\right]\\&+\tfrac{1}{2}I^E_{x_bx_b}\omega^2_{E_1}+\tfrac{1}{2}I^E_{x_bx_b}\omega^2_{E_2}+I^E_{x_bx_b}\omega_{E_1}\omega_1+I^E_{x_bx_b}\omega_{E_2}\omega_1\end{aligned}}$$

Or,

$$\boxed{\begin{aligned}K=\tfrac{1}{2}m_T\left(\dot{X}^2+\dot{Y}^2+\dot{Z}^2\right)&+\tfrac{1}{2}\left[I^G_{x_bx_b}\omega_1^2+I^G_{y_by_b}\omega_2^2+I^G_{z_bz_b}\omega_3^2-2I^G_{x_bz_b}\omega_1\,\omega_3\right]\\&+\tfrac{1}{2}I^E_{x_bx_b}\omega^2_{E_1}+\tfrac{1}{2}I^E_{x_bx_b}\omega^2_{E_2}+I^E_{x_bx_b}\omega_{E_1}\omega_1+I^E_{x_bx_b}\omega_{E_2}\omega_1\end{aligned}}$$

Note (as before) that $I^G_{ij}\ (i,j=x_b,y_b\text{ or }z_b)$ represent moments and products of inertia of the ***entire aircraft*** about its mass center G while $I^E_{x_bx_b}$ represents the moments of inertia of just the ***rotating components*** of the engines about their axes of rotation. Also, recall that although both engines are ***identical***, they may be rotating at ***different speeds***.

Example 5: Double Pendulum or Arm



The system shown is a ***three-dimensional double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ***ball and socket*** joints at O and A. The ***orientation*** of each link is defined relative to the ground using a 3-1-3 ***body-fixed*** rotation sequence. The ***lengths*** of the links are ℓ_1 and ℓ_2. The links are assumed to be ***slender bars*** with mass centers at their ***midpoints***.

Reference frames:

$R : \underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3$ (fixed frame)

$L_i : \underset{\sim}{n}_1^i, \underset{\sim}{n}_2^i, \underset{\sim}{n}_3^i \;\; (i=1,2)$ (fixed in the two links)

Find:

a) $\underset{\sim}{H}_{G_i} \; (i=1,2)$ the angular momenta of the two bars about their respective mass centers

b) $K_i \; (i=1,2)$ the kinetic energies of the bars

Solution:

a) The inertia matrices of the links about the link-fixed directions can be written as

$$\left[I_{G_i}\right]_{L_i} = \tfrac{1}{12} m_i \ell_i^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (i=1,2)$$

In Volume I, Exercise 5.2 it was found that for a 3-1-3 body fixed orientation angle sequence, the ***angular velocity vectors*** of the links can be written as follows.

$${}^R\underset{\sim}{\omega}_{L_i} = \omega_{i1}\, \underset{\sim}{n}_1^i + \omega_{i2}\, \underset{\sim}{n}_2^i + \omega_{i3}\, \underset{\sim}{n}_3^i = \left(\dot{\theta}_{i1} S_{i2} S_{i3} + \dot{\theta}_{i2} C_{i3}\right)\underset{\sim}{n}_1^i + \left(\dot{\theta}_{i1} S_{i2} C_{i3} - \dot{\theta}_{i2} S_{i3}\right)\underset{\sim}{n}_2^i + \left(\dot{\theta}_{i3} + \dot{\theta}_{i1} C_{i2}\right)\underset{\sim}{n}_3^i$$

The ***body-fixed components*** of the ***angular momenta*** of the links can then written as

$$\begin{Bmatrix} \underset{\sim}{H}_{G_i} \cdot \underset{\sim}{n}_1^i \\ \underset{\sim}{H}_{G_i} \cdot \underset{\sim}{n}_2^i \\ \underset{\sim}{H}_{G_i} \cdot \underset{\sim}{n}_3^i \end{Bmatrix} = \tfrac{1}{12} m_i \ell_i^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega_{i1} \\ \omega_{i2} \\ \omega_{i3} \end{Bmatrix} = \tfrac{1}{12} m_i \ell_i^2 \begin{Bmatrix} \omega_{i1} \\ 0 \\ \omega_{i3} \end{Bmatrix}$$

$$\Rightarrow \boxed{\underset{\sim}{H}_{G_i} = \tfrac{1}{12} m_i \ell_i^2 \left(\omega_{i1}\, \underset{\sim}{n}_1^i + \omega_{i3}\, \underset{\sim}{n}_3^i\right)} \qquad (i=1,2)$$

b) The ***kinetic energies*** of the bars must include the ***translational*** and ***rotational energies***. However, since link OA is rotating about a ***fixed-point*** O, the kinetic energy can also be written as ***purely rotational energy*** about point O.

Link 1:

Taking advantage of the fact that link 1 is rotating about a fixed point, first find the angular momentum about the fixed point. Using the parallel axes theorem (or inertia tables directly) to find the inertias about the end of the link gives

$$\begin{Bmatrix} \underset{\sim}{H}_O \cdot \underset{\sim}{n}_1^1 \\ \underset{\sim}{H}_O \cdot \underset{\sim}{n}_2^1 \\ \underset{\sim}{H}_O \cdot \underset{\sim}{n}_3^1 \end{Bmatrix} = \tfrac{1}{3} m_1 \ell_1^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{13} \end{Bmatrix} = \tfrac{1}{3} m_1 \ell_1^2 \begin{Bmatrix} \omega_{11} \\ 0 \\ \omega_{13} \end{Bmatrix} \quad \Rightarrow \boxed{\underset{\sim}{H}_O = \tfrac{1}{3} m_1 \ell_1^2 \left(\omega_{11}\ \underset{\sim}{n}_1^1 + \omega_{13}\ \underset{\sim}{n}_3^1 \right)}$$

Then the kinetic energy can be written as

$$\boxed{K_1 = \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_{L_1} \cdot \underset{\sim}{H}_O = \tfrac{1}{2}\left(\tfrac{1}{3} m_1 \ell_1^2\right)\left(\omega_{11}^2 + \omega_{13}^2\right) = \tfrac{1}{6} m_1 \ell_1^2 \left(\omega_{11}^2 + \omega_{13}^2\right)}$$

This result can also be found using the more general form for the kinetic energy. Using this approach, first find the velocity of the mass center of the link.

$${}^R\underset{\sim}{v}_{G_1} = \underbrace{{}^R\underset{\sim}{v}_O}_{\text{zero}} + {}^R\underset{\sim}{v}_{G_1/O} = {}^R\underset{\sim}{\omega}_{L_1} \times \underset{\sim}{r}_{G_1/O} = \left(\omega_{11}\ \underset{\sim}{n}_1^1 + \omega_{12}\ \underset{\sim}{n}_2^1 + \omega_{13}\ \underset{\sim}{n}_3^1\right) \times \left(-\frac{\ell_1}{2} \underset{\sim}{n}_2^1\right) = -\frac{\ell_1}{2}\left(\omega_{11}\, \underset{\sim}{n}_3^1 - \omega_{13}\, \underset{\sim}{n}_1^1\right)$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{v}_{G_1} = \frac{\ell_1}{2}\left(\omega_{13}\, \underset{\sim}{n}_1^1 - \omega_{11}\, \underset{\sim}{n}_3^1\right)}$$

Using the general form for kinetic energy gives the same result

$$K = \tfrac{1}{2} m_1 \left({}^R\underset{\sim}{v}_{G_1}\right)^2 + \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_{L_1} \cdot \underset{\sim}{H}_{G_1}$$

$$= \tfrac{1}{2} m_1 \left(\tfrac{\ell_1}{2}\right)^2 \left(\omega_{13}^2 + \omega_{11}^2\right) + \tfrac{1}{2}\left(\tfrac{1}{12} m_1 \ell_1^2\right)\left(\omega_{11}^2 + \omega_{13}^2\right) = \left(\tfrac{1}{8} + \tfrac{1}{24}\right) m_1 \ell_1^2 \left(\omega_{11}^2 + \omega_{13}^2\right)$$

$$\Rightarrow \boxed{K_1 = \tfrac{1}{6} m_1 \ell_1^2 \left(\omega_{11}^2 + \omega_{13}^2\right)}$$

Link 2:

First find the velocity of the mass center of the link.

$${}^R\underset{\sim}{v}_{G_2} = {}^R\underset{\sim}{v}_A + {}^R\underset{\sim}{v}_{G_2/A} = \left({}^R\underset{\sim}{\omega}_{L_1} \times \underset{\sim}{r}_{A/O}\right) + \left({}^R\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{r}_{G_2/A}\right)$$

$$= \left(\omega_{11}\ \underset{\sim}{n}_1^1 + \omega_{12}\ \underset{\sim}{n}_2^1 + \omega_{13}\ \underset{\sim}{n}_3^1\right) \times \left(-\ell_1 \underset{\sim}{n}_2^1\right) + \left(\omega_{21}\ \underset{\sim}{n}_1^2 + \omega_{22}\ \underset{\sim}{n}_2^2 + \omega_{23}\ \underset{\sim}{n}_3^2\right) \times \left(-\frac{\ell_2}{2} \underset{\sim}{n}_2^2\right)$$

$$= -\ell_1 \left(\omega_{11}\, \underset{\sim}{n}_3^1 - \omega_{13}\, \underset{\sim}{n}_1^1\right) - \frac{\ell_2}{2}\left(\omega_{21}\, \underset{\sim}{n}_3^2 - \omega_{23}\, \underset{\sim}{n}_1^2\right)$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{v}_{G_2} = \ell_1 \left(\omega_{13}\, \underset{\sim}{n}_1^1 - \omega_{11}\, \underset{\sim}{n}_3^1\right) + \frac{\ell_2}{2}\left(\omega_{23}\, \underset{\sim}{n}_1^2 - \omega_{21}\, \underset{\sim}{n}_3^2\right)}$$

Transformation matrices can now be used to resolve all the components of ${}^R\underset{\sim}{v}_{G_2}$ into the base frame.

$$
{}^{R}\underset{\sim}{v}_{G_2} = \ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1^1 \\ \underset{\sim}{n}_2^1 \\ \underset{\sim}{n}_3^1 \end{Bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1^2 \\ \underset{\sim}{n}_2^2 \\ \underset{\sim}{n}_3^2 \end{Bmatrix}
$$

$$
= \ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}
$$

So, the components of ${}^{R}\underset{\sim}{v}_{G_2}$ in the base system are

$$
\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}^T = \left[\ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix} \right]^T
$$

$$
= \left[\ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} \right]^T + \left[\frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix} \right]^T
$$

$$
\Rightarrow \boxed{ \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{Bmatrix} \omega_{13} \\ 0 \\ -\omega_{11} \end{Bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} R_2 \end{bmatrix}^T \begin{Bmatrix} \omega_{23} \\ 0 \\ -\omega_{21} \end{Bmatrix} }
$$

The kinetic energy of link 2 can now be written as

$$
K_2 = \tfrac{1}{2} m_2 ({}^{R}\underset{\sim}{v}_{G_2})^2 + \tfrac{1}{2}\, {}^{R}\underset{\sim}{\omega}_{L_2} \cdot \underset{\sim}{H}_{G_2}
$$

$$
= \tfrac{1}{2} m_2 \left(V_1^2 + V_2^2 + V_3^2 \right) + \tfrac{1}{2} \left(\omega_{21}\ \underset{\sim}{n}_1^2 + \omega_{22}\ \underset{\sim}{n}_2^2 + \omega_{23}\ \underset{\sim}{n}_3^2 \right) \cdot \left(\tfrac{1}{12} m_2 \ell_2^2 \left(\omega_{21}\ \underset{\sim}{n}_1^2 + \omega_{23}\ \underset{\sim}{n}_3^2 \right) \right)
$$

$$
\Rightarrow \boxed{ K_2 = \tfrac{1}{2} m_2 \left(V_1^2 + V_2^2 + V_3^2 \right) + \tfrac{1}{24} m_2 \ell_2^2 \left(\omega_{21}^2 + \omega_{23}^2 \right) }
$$

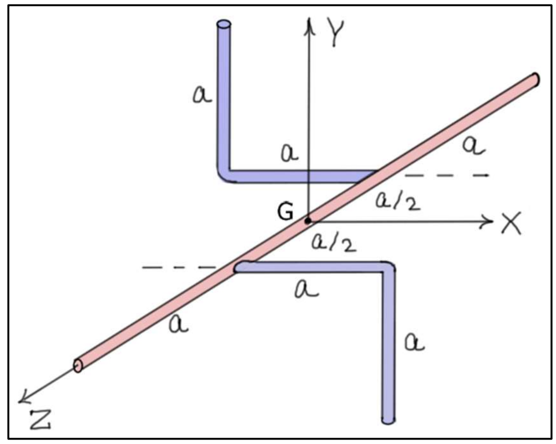
Clearly, the ***most intricate part*** of the kinetic energy of link 2 is in the ***translational energy***.

Exercises

The following observations are helpful in the solution of problems 1.1 and 1.2.

1. If α is an arbitrary scalar, and if λ is an eigenvalue of matrix $[A]$, then $\alpha\lambda$ is an eigenvalue of matrix $\alpha[A]$.
2. If α is an arbitrary scalar, and if $\underset{\sim}{x}$ is an eigenvector of matrix $[A]$, then $\underset{\sim}{x}$ is also an eigenvector of matrix $\alpha[A]$ corresponding to the eigenvalue $\alpha\lambda$.

1.1 The body shown consists of two L-shaped arms welded to a straight rod. The straight segment has length $3a$, and each segment of the L-shaped arms has length a. Each segment of length a has mass m. All segments are slender.



a) Find the ***principal moments*** of ***inertia*** and the ***principal directions*** for the mass-canter G.

b) ***Show*** that the eigenvector (or modal) matrix found in part (a) diagonalizes the inertia matrix.

Answers:

$$[I_G] = ma^2\begin{bmatrix} \frac{47}{12} & 1 & -\frac{3}{2} \\ 1 & \frac{71}{12} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{10}{3} \end{bmatrix} \approx ma^2\begin{bmatrix} 3.91667 & 1.00000 & -1.50000 \\ 1.00000 & 5.91667 & 0.500000 \\ -1.50000 & 0.500000 & 3.33333 \end{bmatrix}$$

$$[M] \approx \begin{bmatrix} 0.642082 & -0.648393 & -0.409045 \\ -0.246360 & 0.330748 & -0.910995 \\ 0.725974 & 0.685706 & 0.052629 \end{bmatrix} \qquad \det[M] = 1$$

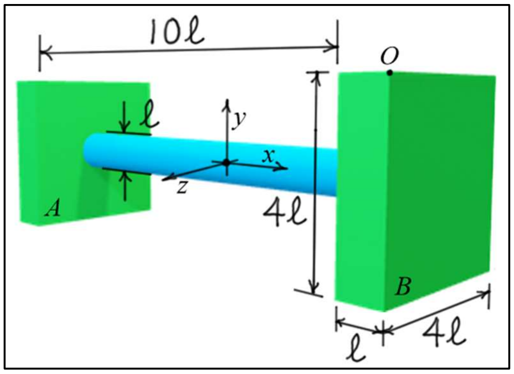
$$I_1 \approx 1.83699\, m\, a^2 \qquad I_2 \approx 4.99288\, m\, a^2 \qquad I_3 \approx 6.33679\, m\, a^2$$

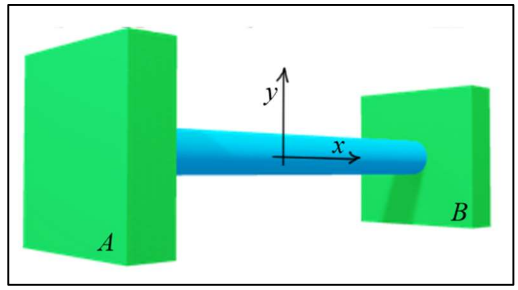
$$\underset{\sim}{e}_1 = 0.642082\, \underset{\sim}{N}_1 - 0.246360\, \underset{\sim}{N}_2 + 0.725974\, \underset{\sim}{N}_3$$

$$\underset{\sim}{e}_2 = -0.648393\, \underset{\sim}{N}_1 + 0.330748\, \underset{\sim}{N}_2 + 0.685706\, \underset{\sim}{N}_3$$

$$\underset{\sim}{e}_3 = -0.409045\, \underset{\sim}{N}_1 - 0.910995\, \underset{\sim}{N}_2 + 0.052629\, \underset{\sim}{N}_3$$

1.2 The figures below show two views of a body with a central cylindrical section and two identical, box-like ends. The central cylindrical section has a diameter of ℓ and length of 10ℓ. The box-like ends have two square sides (length and width equal to 4ℓ) and a depth of ℓ. The cylinder has mass m and the box-like ends each have mass $2m$, so the total mass of the composite shape is $5m$. Find the ***principal moments*** of ***inertia*** and the ***principal directions*** for the point O on the outer corner of end B.





Answers:

$$[I_O] = m\ell^2 \begin{bmatrix} \frac{1219}{24} & -60 & -60 \\ -60 & \frac{5361}{16} & -20 \\ -60 & -20 & \frac{5361}{16} \end{bmatrix} \approx m\ell^2 \begin{bmatrix} 50.792 & -60 & -60 \\ -60 & 335.06 & -20 \\ -60 & -20 & 335.06 \end{bmatrix}$$

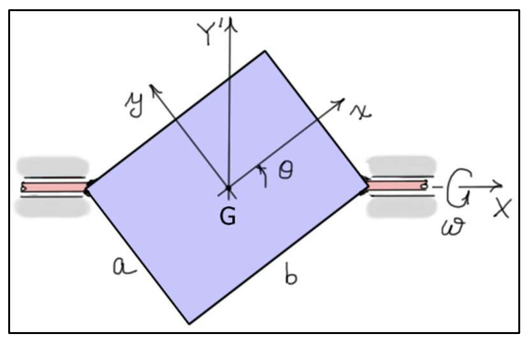
$$[M] \approx \begin{bmatrix} 0.959542 & -0.281564 & 0.000000 \\ 0.199096 & 0.678499 & -0.707107 \\ 0.199096 & 0.678499 & 0.707107 \end{bmatrix} \qquad \det[M] = 1$$

$$I_1 \approx 25.8928\, m\ell^2 \qquad I_2 \approx 339.961\, m\ell^2 \qquad I_3 = 355.062\, m\ell^2$$

$$\underset{\sim}{e}_1 = 0.959542\, \underset{\sim}{N}_1 + 0.199096\, \underset{\sim}{N}_2 + 0.199096\, \underset{\sim}{N}_3$$

$$\underset{\sim}{e}_2 = -0.281564\, \underset{\sim}{N}_1 + 0.678499\, \underset{\sim}{N}_2 + 0.678499\, \underset{\sim}{N}_3 \qquad \underset{\sim}{e}_3 = -0.707107\, \underset{\sim}{N}_2 + 0.707107\, \underset{\sim}{N}_3$$

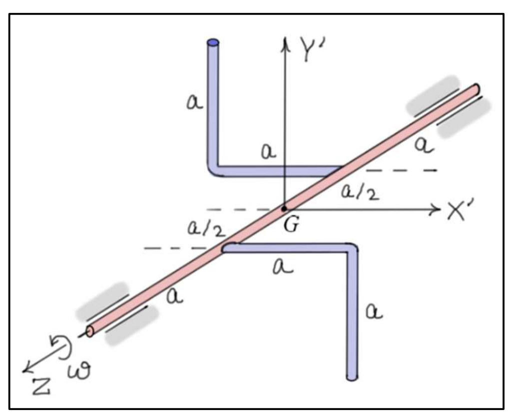
1.3 The rectangular plate P is welded to a shaft so that it rotates about its diagonal. (a) Find $\underset{\sim}{H}_G$ the angular momentum of P about its mass center G. Express your results in the X, Y′, and Z′ directions. (b) Find K the kinetic energy of the plate.



Answers:

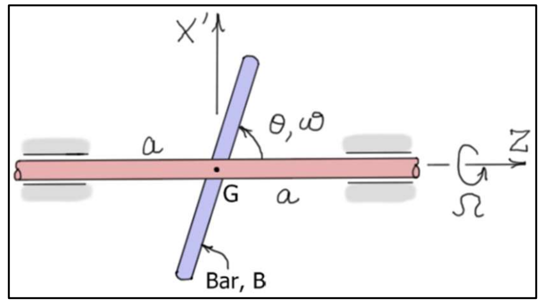
a) $\underset{\sim}{H}_G = \dfrac{m\,a\,b\,\omega}{12(a^2+b^2)}\left(2\,a\,b\,\underset{\sim}{i} + (a^2-b^2)\,\underset{\sim}{j}'\right)$ b) $K = ma^2b^2\omega^2 / 12(a^2+b^2)$

1.4 The system shown consists of two L-shaped arms welded to a shaft of length $3a$. The planes of the arms are at right angles to the shaft. If all parts are made of "slender" bars, complete the following. (a) Find $\underset{\sim}{H}_G$ the angular momentum of the system about its mass center G. Express your results in the X′, Y′, and Z directions. (b) Find K the kinetic energy of the system.



Answers: a) $\underset{\sim}{H}_G = ma^2\omega\left(-\tfrac{3}{2}\,\underset{\sim}{i}' + \tfrac{1}{2}\,\underset{\sim}{j}' + \tfrac{10}{3}\,\underset{\sim}{k}\right)$ b) $K = \tfrac{5}{3}ma^2\omega^2$

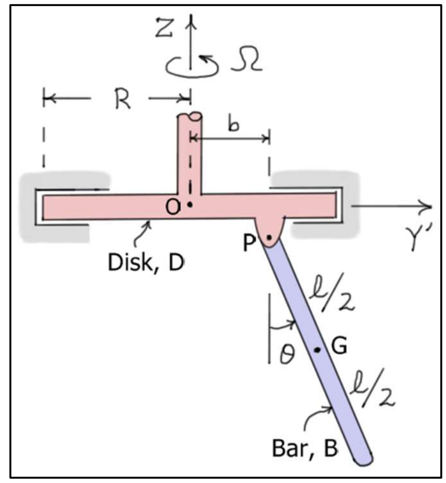
1.5 The system shown consists of a bar B that is pinned through the center of a shaft of length $2a$. As the shaft rotates about the Z-axis at a rate Ω (r/s), B rotates about the Y' at a rate $\dot{\theta} = \omega$ (r/s). (a) Find $\underset{\sim}{H}_G$ the angular momentum of B about its mass center G. Express your results in the X', Y', and Z directions. (b) Find K the kinetic energy of B.



Answers:

a) $\underset{\sim}{H}_G = \frac{1}{12} m\ell^2 \left(-\left(\Omega S_\theta C_\theta\right) \underset{\sim}{i}' + \omega \underset{\sim}{j}' + \left(\Omega S_\theta^2\right) \underset{\sim}{k} \right)$ b) $K = \frac{1}{24} m\ell^2 \left(\omega^2 + \Omega^2 S_\theta^2 \right)$

1.6 The system shown consists of a bar B that is pinned to the bottom of a disk D. As the disk rotates at a rate Ω (rad/sec) about the Z-axis, the bar rotates at a rate $\dot{\theta}$ (rad/sec) about the X' direction. (a) Find $\underset{\sim}{H}_G$ the angular momentum of B about its mass center G. Express your results in the X', Y', and Z directions. (b) Find K the kinetic energy of B.

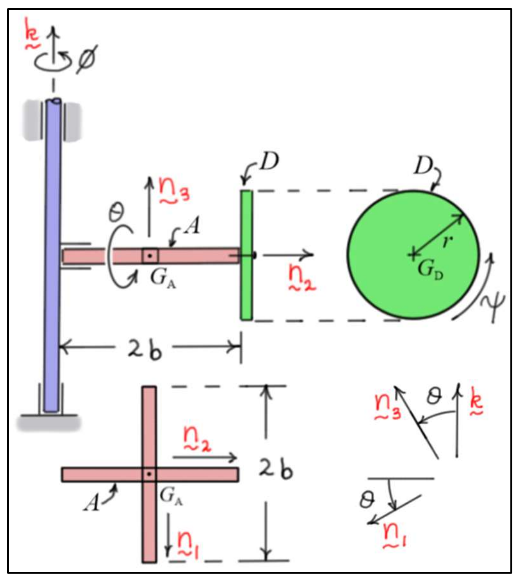


Answers:

a) $\underset{\sim}{H}_G = \frac{1}{12} m\ell^2 \left(\dot{\theta}\, \underset{\sim}{i}' + \left(\Omega S_\theta C_\theta\right) \underset{\sim}{j}' + \left(\Omega S_\theta^2\right) \underset{\sim}{k} \right)$

b) $K = \frac{1}{2} m \left(b\Omega + \frac{1}{2} \ell \Omega S_\theta \right)^2 + \frac{1}{6} m\ell^2 \dot{\theta}^2 + \frac{1}{24} m\ell^2 \Omega^2 S_\theta^2$

1.7 The system shown consists of two bodies, the cross-shaped frame A and the disk D. Frame A is connected to the ground with a two-axis joint whose motion is described by the angles ϕ and θ. The angle ϕ allows A to rotate about a vertical axis while the angle θ allows an additional rotation about the rotating $\underset{\sim}{n}_2$ direction. Disk D is pinned to the end of A and can rotate relative to A also about the $\underset{\sim}{n}_2$ direction. The unit vector set $A : (\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ are fixed in the frame A. The points G_A and G_D represent the mass centers of A and D. a) Find $\underset{\sim}{H}_{G_A}$ and $\underset{\sim}{H}_{G_D}$ the angular momenta of A and D about their mass centers, and b) Find K the kinetic energy of the system. The system mass center G lies a distance d_A to the right of G_A and a distance d_D to the left of G_D.



Answers:

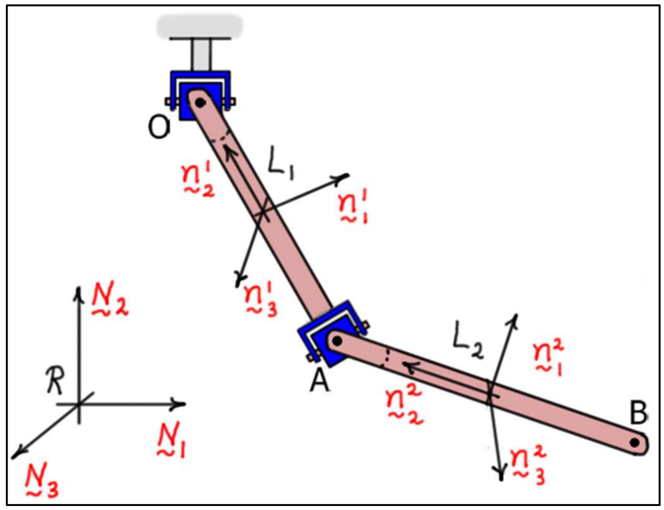
a) $\underset{\sim}{H}_{G_A} = I_{11}^{G_A} \omega_1 \underset{\sim}{n}_1 + I_{22}^{G_A} \omega_2 \underset{\sim}{n}_2 + I_{33}^{G_A} \omega_3 \underset{\sim}{n}_3$ $\underset{\sim}{H}_{G_D} = I_{11}^{G_D} \omega_1 \underset{\sim}{n}_1 + I_{22}^{G_D} \left(\omega_2 + \omega_D \right) \underset{\sim}{n}_2 + I_{33}^{G_D} \omega_3 \underset{\sim}{n}_3$

b) $$K = \tfrac{1}{2}\left(m_A + 4m_D\right)b^2\left(\omega_1^2 + \omega_3^2\right) + \tfrac{1}{2}\left[I_{11}^{G_A}\,\omega_1^2 + I_{22}^{G_A}\,\omega_2^2 + I_{33}^{G_A}\,\omega_3^2\right] + \tfrac{1}{2}\left[I_{11}^{G_D}\,\omega_1^2 + I_{22}^{G_D}\left(\omega_2 + \omega_D\right)^2 + I_{33}^{G_D}\,\omega_3^2\right]$$

or

$$K = \tfrac{1}{2}m_T\left(b + d_A\right)^2\left(\omega_1^2 + \omega_3^2\right) + \tfrac{1}{2}\left[\left(I_{11}^G\right)_{A+D}\omega_1^2 + \left(I_{22}^G\right)_{A+D}\omega_2^2 + \left(I_{33}^G\right)_{A+D}\omega_3^2\right] + \tfrac{1}{2}\left(I_{22}^G\right)_D\left(\omega_D^2 + 2\omega_2\omega_D\right)$$

1.8 The system shown is a ***three-dimensional double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ***universal joints*** at O and A, respectively. The ground frame is $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ and the link frames are $L_i:\left(\underset{\sim}{n}_1^i, \underset{\sim}{n}_2^i, \underset{\sim}{n}_3^i\right)$ $(i = 1,2)$. The ***orientation*** of L_1 is defined ***relative*** to R and the orientation of L_2 is defined ***relative*** to L_1 each with a 1-3 ***body-fixed*** rotation sequence.



Link OA is oriented relative to the ground frame by first rotating through an angle θ_{11} about the $\underset{\sim}{N}_1$ direction, and then rotating about an angle θ_{12} about the $\underset{\sim}{n}_3^1$ direction. Link AB is oriented relative to link OA by rotating first through an angle θ_{21} about the $\underset{\sim}{n}_1^1$ direction, and then through an angle θ_{22} about the $\underset{\sim}{n}_3^2$ direction. The lengths of the links are ℓ_1 and ℓ_2 with mass centers are at their midpoints. Find a) $\underset{\sim}{H}_{G_i}\,(i = 1,2)$ the angular momenta of the two bars about their respective mass centers, and b) $K_i\,(i = 1,2)$ the kinetic energies of the bars.

Answers:

$${}^R\underset{\sim}{\omega}_{L_1} = \omega_{11}\,\underset{\sim}{n}_1^1 + \omega_{12}\,\underset{\sim}{n}_2^1 + \omega_{13}\,\underset{\sim}{n}_3^1 = \dot{\theta}_{11}\,C_{12}\,\underset{\sim}{n}_1^1 - \dot{\theta}_{11}S_{12}\,\underset{\sim}{n}_2^1 + \dot{\theta}_{12}\,\underset{\sim}{n}_3^1$$

$${}^R\underset{\sim}{\omega}_{L_2} = {}^R\underset{\sim}{\omega}_{L_1} + {}^{L_1}\underset{\sim}{\omega}_{L_2} = \omega_{21}\,\underset{\sim}{n}_1^2 + \omega_{22}\,\underset{\sim}{n}_2^2 + \omega_{23}\,\underset{\sim}{n}_3^2$$

$$\Rightarrow \begin{Bmatrix}\omega_{21}\\ \omega_{22}\\ \omega_{23}\end{Bmatrix} = \left[R_2\right]\begin{Bmatrix}C_{12}\,\dot{\theta}_{11}\\ -S_{12}\,\dot{\theta}_{11}\\ \dot{\theta}_{12}\end{Bmatrix} + \begin{Bmatrix}C_{22}\,\dot{\theta}_{21}\\ -S_{22}\,\dot{\theta}_{21}\\ \dot{\theta}_{22}\end{Bmatrix}$$

$$\boxed{\underset{\sim}{H}_{G_i} = \tfrac{1}{12} m_i \, \ell_i^2 \left(\omega_{i1} \; \underset{\sim}{n}_1^i + \omega_{i3} \; \underset{\sim}{n}_3^i \right)}$$

$$\boxed{K_{L_1} = \tfrac{1}{6} m_1 \, \ell_1^2 \left(\omega_{11}^2 + \omega_{13}^2 \right)} \quad \boxed{K_{L_2} = \tfrac{1}{2} m_2 \left(V_1^2 + V_2^2 + V_3^2 \right) + \tfrac{1}{24} m_2 \, \ell_2^2 \left(\omega_{21}^2 + \omega_{23}^2 \right)}$$

$$\boxed{\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \ell_1 \left[R_1 \right]^T \begin{Bmatrix} \omega_{13} \\ 0 \\ -\omega_{11} \end{Bmatrix} + \frac{\ell_2}{2} \left[R_1 \right]^T \left[R_2 \right]^T \begin{Bmatrix} \omega_{23} \\ 0 \\ -\omega_{21} \end{Bmatrix}} \quad \boxed{\left[R_i \right] = \begin{bmatrix} C_{i2} & C_{i1} S_{i2} & S_{i1} S_{i2} \\ -S_{i2} & C_{i1} C_{i2} & S_{i1} C_{i2} \\ 0 & -S_{i1} & C_{i1} \end{bmatrix}}$$

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Addendum on Inertia – Non-distinct Eigenvalues

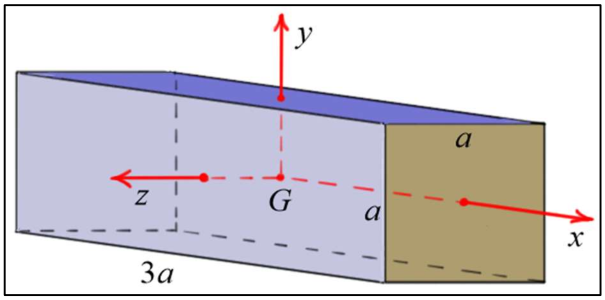
It was noted above that ***nonsymmetrical bodies*** can have an ***infinite number*** of ***inertia matrices*** associated with ***each point*** in the body, because the inertia matrix changes with the orientation of the axes at that point. All those inertia matrices can generally be reduced to a ***single unique*** inertia matrix about a ***unique*** set of ***principal axes*** at that point. However, for this to be true, the ***principal moments*** of ***inertia*** must be ***distinct*** (i.e., not equal).

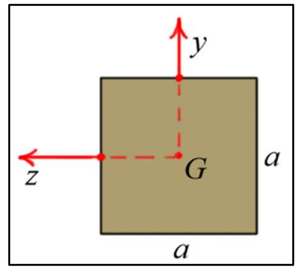
Symmetrical bodies also have a unique set of principal moments of inertia at any point. However, symmetrical bodies can have ***multiple sets*** of ***principal axes*** at a given point and ***multiple points*** can have the ***same principal moments*** of ***inertia*** and ***principal axes***. A ***body of revolution*** was used above as an example.

It turns out that a point of a body has ***multiple sets*** of ***principal axes*** whenever all the ***eigenvalues*** of the inertia matrix for that point are ***not distinct***, that is whenever two or all three of the eigenvalues are equal. The body ***may be*** symmetrical about the plane formed by the eigenvectors, but it may not.

Example 6: Square Prism

To illustrate this situation, consider the square prism shown below having square ends with sides of length a and a total prism length of $3a$. The mass center G is at the center of the prism and the axes of the reference frame $B:(x, y, z)$ located at G are perpendicular to and pass through the centers of the sides as shown.





The *x-y*, *x-z*, and *y-z* planes are all planes of symmetry, so the three axes are principal axes and the moments of inertia about these axes are the principal moments of inertia. Using a set of ***standard inertia tables***, the principal inertias and inertia matrix can be written as follows.

$$I_{xx} = \tfrac{1}{12} m\left(a^2 + a^2\right) = \tfrac{1}{6} ma^2 \qquad I_{yy} = I_{zz} = \tfrac{1}{12} m\left(a^2 + (3a)^2\right) = \tfrac{10}{12} ma^2 = \tfrac{5}{6} ma^2$$

$$\left[I_G^B\right] = \frac{ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Now consider rotating frame B relative to a frame $C:(X, Y, Z)$ using a 1-2-3 body-fixed rotation sequence as defined in Unit 5 of Volume I. The transformation matrix that transforms vector components from C into B can be written as follows.

$$\begin{aligned}[R] &= [R_3][R_2][R_1] \\ &= \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & S_1 \\ 0 & -S_1 & C_1 \end{bmatrix} = \begin{bmatrix} C_2C_3 & C_1S_3 + S_1S_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix}\end{aligned}$$

Here, $S_i\ (i = 1, 2, 3)$ and $C_i\ (i = 1, 2, 3)$ represent the sines and cosines of orientation angles θ_1, θ_2, and θ_3. Using results presented earlier in this unit, the inertia matrix about the axes of frame C can be calculated as follows.

$$\left[I_G^C\right] = [R]^T \left[I_G^B\right][R]$$

As an example, consider rotating frame B relative to C using the angles $\theta_1 = 20$ (deg), $\theta_2 = -30$ (deg), and $\theta_3 = 60$ (deg). Using these values, the transformation matrix $[R]$ is found to be approximately

$$[R] \approx \begin{bmatrix} 0.433013 & 0.728293 & 0.531121 \\ -0.75 & 0.617945 & -0.235889 \\ -0.5 & -0.296198 & 0.813798 \end{bmatrix}$$

Using this transformation matrix, the inertia matrix about the axes of reference frame C is found to be

$$\left[I_G^C\right] = [R]^T \left[I_G^B\right][R] \approx \frac{ma^2}{6} \begin{bmatrix} 4.25 & -1.26144 & -0.91993 \\ -1.26144 & 2.87836 & -1.54725 \\ -0.91993 & -1.54725 & 3.87164 \end{bmatrix}$$

In theory, this process can now be reversed by calculating the eigenvalues and eigenvectors of $\left[I_G^C\right]$.

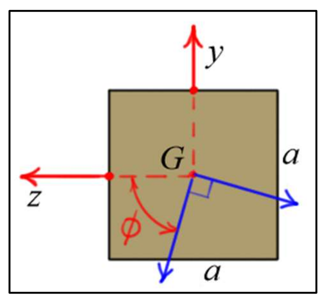
As noted in the Exercises, if α is an arbitrary scalar, and if λ is an eigenvalue of matrix $[A]$, then $\alpha\lambda$ is an eigenvalue of $\alpha[A]$. Also, if $\underset{\sim}{x}$ is an eigenvector of matrix $[A]$, the $\underset{\sim}{x}$ is also an eigenvector of matrix $\alpha[A]$. So, if we define the matrix $[A]$ as

$$[A] = \begin{bmatrix} 4.25 & -1.26144 & -0.91993 \\ -1.26144 & 2.87836 & -1.54725 \\ -0.91993 & -1.54725 & 3.87164 \end{bmatrix}$$

then the eigenvalues of $\left[I_G^C\right]$ are $\dfrac{ma}{6}$ times the eigenvalues of $[A]$. Also, the eigenvectors of $\left[I_G^C\right]$ are the same as the eigenvectors of $[A]$.

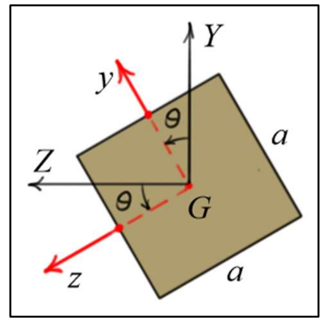
Using MATLAB, the eigenvalues and eigenvectors of $\left[I_G^C\right]$ are found to be approximately

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} \approx \frac{ma^2}{6} \begin{Bmatrix} 1 \\ 5 \\ 5 \end{Bmatrix} \qquad [M]_B \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.960465 & -0.278402 \\ 0 & -0.278402 & 0.960465 \end{bmatrix} \qquad [M]_C \approx \begin{bmatrix} 0.43301 & 0.85955 & -0.27143 \\ 0.72829 & -0.51105 & -0.45653 \\ 0.53112 & 0 & 0.84730 \end{bmatrix}$$



Here, the components of the eigenvectors in frame B form the columns of the matrix $[M]_B$ and the components of the eigenvectors in frame C form the columns of the matrix $[M]_C$. Notice the first eigenvector (first column of $[M]_B$) points along the x axis as we expect. However, the second and third do not point along the y and z axes. In fact, the second and third eigenvectors are perpendicular to each other but have both y and z components. The second eigenvector makes an angle $\phi \approx 73.8 \text{ (deg)}$ with the z axis. See the pair of blue vectors in the diagram.

Are these principal axes? Certainly, none of the planes associated with the x axis and these two eigenvectors are planes of symmetry. To answer this question, consider rotating frame B relative to frame C using a single rotation through an angle θ about the x axis. In this case, the transformation matrix that transforms vector components from C into B is



$$\boxed{[R]=[R_1]=\begin{bmatrix}1 & 0 & 0\\ 0 & C_1 & S_1\\ 0 & -S_1 & C_1\end{bmatrix}}$$

The inertia matrix about the axes of frame C can then be calculated as follows.

$$\left[I_G^C\right]=[R_1]^T\left[I_G^B\right][R_1]=\frac{ma^2}{6}\begin{bmatrix}1 & 0 & 0\\ 0 & C_1 & -S_1\\ 0 & S_1 & C_1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & C_1 & S_1\\ 0 & -S_1 & C_1\end{bmatrix}$$

$$=\frac{ma^2}{6}\begin{bmatrix}1 & 0 & 0\\ 0 & C_1 & -S_1\\ 0 & S_1 & C_1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 5C_1 & 5S_1\\ 0 & -5S_1 & 5C_1\end{bmatrix}$$

$$=\frac{ma^2}{6}\begin{bmatrix}1 & 0 & 0\\ 0 & 5\left(C_1^2+S_1^2\right) & 5C_1S_1-5C_1S_1\\ 0 & 5C_1S_1-5C_1S_1 & 5\left(S_1^2+C_1^2\right)\end{bmatrix}$$

$$\Rightarrow \boxed{\left[I_G^C\right]=\frac{ma^2}{6}\begin{bmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}=\left[I_G^B\right]}$$

So, the inertia matrix remains unchanged by an arbitrary rotation about the x axis.

As a final step in this example, we can make sure the eigenvector matrix can be used to diagonalize the inertia matrix. For this we need the eigenvectors expressed in the frame C.

$$\left[I_G^C\right][M]_C \approx \frac{ma^2}{6}\begin{bmatrix}4.25 & -1.26144 & -0.91993\\ -1.26144 & 2.87836 & -1.54725\\ -0.91993 & -1.54725 & 3.87164\end{bmatrix}\begin{bmatrix}0.43301 & 0.85955 & -0.27143\\ 0.72829 & -0.51105 & -0.45653\\ 0.53112 & 0 & 0.84730\end{bmatrix}$$

$$\approx \frac{ma^2}{6}\begin{bmatrix}0.43301 & 4.29775 & -1.35715\\ 0.72829 & -2.55526 & -2.28263\\ 0.53112 & 0 & 4.23648\end{bmatrix}$$

$$[M]_C^T\left[I_G^C\right][M]_C \approx \frac{ma^2}{6}\begin{bmatrix} 0.43301 & 0.72829 & 0.53112 \\ 0.85955 & -0.51105 & 0 \\ -0.27143 & -0.45653 & 0.84730 \end{bmatrix}\begin{bmatrix} 0.43301 & 4.29775 & -1.35715 \\ 0.72829 & -2.55526 & -2.28263 \\ 0.53112 & 0 & 4.23648 \end{bmatrix}$$

$$\approx \frac{ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

So, in fact, the eigenvector matrix can be used to diagonalize the inertia matrix.

Final Note

Although the above result was derived ***specifically*** for the ***square prism***, the results are true so long as the ***inertias*** about ***two*** of the ***principal axes*** are ***equal***. Hence, for a body, if ***two*** of the ***principal inertias*** are ***equal***, then – 1) ***any axis perpendicular*** to the axis associated with the ***third*** (distinct) ***principal inertia*** is a ***principal axis***, and 2) an ***arbitrary rotation*** about the axis associated with the ***third*** (distinct) ***principal inertia does not change*** the ***inertia matrix***. This result can be ***extended*** to show that ***all vectors*** are ***eigenvectors*** if ***all three principal inertias*** are ***equal***.