An Introduction to Three-Dimensional, Rigid Body Dynamics

James W. Kamman, PhD

Volume II: Kinetics

Unit 5

d'Alembert's Principle and Kane's Equations

Summary

This unit discusses the use of d'Alembert's principle and Kane's equations to develop the equations of motion of rigid body dynamic systems. The application of these methods relies heavily on the concepts of ***degrees of freedom***, ***generalized coordinates***, ***partial velocities***, ***partial angular velocities,*** and ***generalized forces*** discussed in Unit 3. As discussed in Units 3 and 4, systems with multiple bodies will be analyzed as ***systems*** rather than one body at a time as with the Newton/Euler equations of motion. Examples include an aircraft with two engines and an upright, two-wheeled bicycle.

Page Count	Examples	Suggested Exercises
64	7	8

d'Alembert's Principle

Like ***Lagrange's Equations***, the application of ***d'Alembert's principle*** differs from the application of the Newton/Euler equations of motion (EOM) presented in Unit 2 of this volume in the following ways:

- Focus is on the ***entire system*** rather than individual components
- ***Constraint forces*** and ***moments*** that are ***inactive*** are ***eliminated*** from the analysis

Unlike Lagrange's equations, d'Alembert's principle ***does not*** require ***differentiation*** of the system's ***kinetic energy function***. It does, however, require calculation of ***accelerations*** and ***angular accelerations***.

For many systems, the resulting equations of motion form a set of ***differential*** equations. For more complex systems the EOM form a set of ***differential*** and ***algebraic*** equations.

d'Alembert's Principle for Multi-Degree-of-Freedom Systems

Consider a system of N_B rigid bodies with "n" degrees of freedom. Previously, it was noted that the differential equations of motion (DEOM) of a such systems can be written for a set of "n" independent generalized coordinates $q_k \; (k=1,\ldots,n)$ using Lagrange's equations. Another approach to finding the DEOM is to use ***d'Alembert's principle***. For a system of N_B rigid bodies with "n" degrees of freedom moving in an inertial frame R, d'Alembert's principle can be written as

$$\boxed{\sum_{i=1}^{N_B}\left(m_i\,{}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial\dot{q}_k}\right)+\sum_{i=1}^{N_B}\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{B_i}}{\partial\dot{q}_k}=F_{q_k}} \qquad (k=1,\ldots,n)$$

The individual terms in the equations are defined as follows.

$m_i =$ mass of body B_i

${}^R\underset{\sim}{v}_{G_i} =$ velocity of G_i, the mass center of B_i

${}^R\underset{\sim}{a}_{G_i} =$ acceleration of G_i, the mass center of B_i

${}^R\underset{\sim}{\omega}_{B_i} =$ angular velocity of B_i

${}^R\underset{\sim}{\alpha}_{B_i} =$ angular acceleration of B_i

$\underset{\approx}{I}_{G_i} =$ inertia dyadic of B_i for its mass center G_i

$\underset{\sim}{H}_{G_i} =$ angular momentum of B_i about its mass center G_i

$F_{q_k} =$ generalized force associated with generalized coordinate q_k

Notes:

1. The boxed equations represent "*n*" differential equations for the "*n*" independent generalized coordinates $q_k\ (k=1,\ldots,n)$. It is important that all quantities be written ***only*** in terms of $q_k\ (k=1,\ldots,n)$ and their time derivatives. No other variables are permitted.
2. The ***right side*** of the EOM are the ***same*** as for Lagrange's equations. All forces and torques (both conservative and nonconservative) are included in F_{q_k}.
3. As noted above, calculation of the left side of d'Alembert's principle ***does not*** require differentiation of the kinetic energy function, but it ***does*** require calculation of accelerations and angular accelerations.
4. This form of d'Alembert's principle can be used for systems ***without constraints*** or for systems with ***holonomic constraints*** provided it is convenient to use the constraints to ***eliminate surplus generalized coordinates***.
5. For systems with ***nonholonomic constraints*** or for systems with ***holonomic constraints*** for which it is not convenient to eliminate surplus generalized coordinates, ***Lagrange multipliers*** are used. See Unit 6 for applications using Lagrange multipliers.

Kane's Equations for Multi-Degree-of-Freedom Systems

Kane's equations are very much like d'Alembert's principle. The major difference is that Kane's equations are written for (or based on) a set of ***independent generalized speeds*** which are not required to be time derivatives of a set of generalized coordinates. Generalized speeds can, in general, be ***linear combinations*** of time derivatives of the generalized coordinates. This generalization can simplify the equations of motion that are generated.

Consider a system of N_B rigid bodies with "*n*" ***degrees of freedom***. Previously, it was noted that equations of motion of a such systems can also be written for the "*n*" generalized coordinates $q_k\ (k=1,\ldots,n)$ using Lagrange's equations or d'Alembert's principle. An alternate approach is to use ***Kane's equations***. For a system of N_B rigid bodies with "*n*" degrees of freedom moving in an inertial frame *R*, Kane's equations of motion can be written as

$$\boxed{\sum_{i=1}^{N_B}\left(m_i\ {}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right)+\sum_{i=1}^{N_B}\left[\left(\underset{\sim}{I}_{G_i}\cdot{}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}=F_{u_k}}\qquad (k=1,\ldots,n)$$

Here, $u_k\ (k=1,\ldots,n)$ are a set of "*n*" ***independent generalized speeds***. These equations are like d'Alembert's principle except the partial velocities are based on a set of independent generalized speeds and not an independent set of generalized coordinates. The individual terms in the equations are all as defined above for d'Alembert's principle except for the partial velocities and generalized forces.

The terms $\dfrac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k}$ and $\dfrac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}$ are the ***partial velocities*** of the mass centers of the bodies and the ***partial angular velocities*** of the bodies associated with the ***generalized speeds***. The generalized forces are also calculated in terms of the generalized speeds and are defined as follows.

$$F_{u_k} = \sum_i \left(\underset{\sim}{F}_i \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{P_i}}{\partial u_k} \right) + \sum_j \left(\underset{\sim}{M}_j \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{B_j}}{\partial u_k} \right)$$

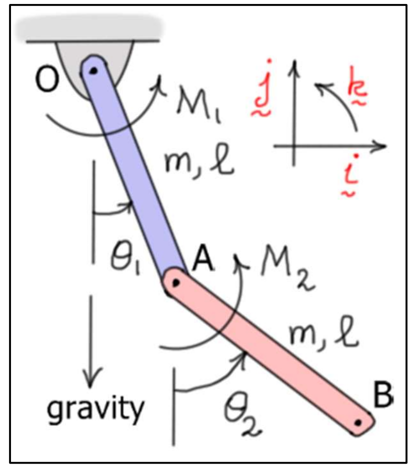
Notes

1. As presented above, ***Kane's equations*** can be written using a set of ***independent*** generalized speeds $u_k\ (k=1,\ldots,n)$ and a set of ***independent*** generalized coordinates $q_k\ (k=1,\ldots,n)$. In this case, Kane's equations along with the ***kinematic differential equations*** represent "2*n*" first-order differential equations for the "*n*" independent generalized coordinates and the "*n*" independent generalized speeds. It is important that all quantities be written only in terms of $q_k\ (k=1,\ldots,n)$ and $u_k\ (k=1,\ldots,n)$. No other variables are allowed.
2. As presented above, Kane's equations can also be written using a set of ***independent*** generalized speeds $u_k\ (k=1,\ldots,n)$ and a set of ***dependent*** generalized coordinates. Note that only the generalized speeds must be independent. In this case, Kane's equations must then be supplemented with ***kinematical differential equations*** and ***differentiated constraint equations*** (relating the dependent generalized coordinates) so the total number of first-order differential equations is equal to the total number of ***independent generalized speeds plus the total number of generalized coordinates***. This characteristic makes Kane's equations more versatile than d'Alembert's principle.
3. The right-hand-side of Kane's equations are ***like*** that for Lagrange's equations and d'Alembert's principle, except that the ***partial velocities*** and ***partial angular velocities*** are defined for the ***generalized speeds***. Of course, if the generalized speeds are defined as the time derivatives of the generalized coordinates, then Kane's equations become the same as d'Alembert's principle. All forces and torques (both conservative and nonconservative) are included in F_{u_k}.

5. If it is more convenient to use a set of ***dependent set of generalized speeds***, then ***Lagrange multipliers*** must be utilized. See Unit 6 for applications using Lagrange multipliers.

Example 1: Two Link, Planar Double Pendulum with Driving Torques

The figure to the right shows a ***double pendulum*** or ***arm*** in a vertical plane with ***driving torques*** at the joints. The two ***uniform slender links*** (L_1, L_2) are assumed to be identical with mass m and length ℓ. The system has ***two degrees of freedom*** described by the ***generalized coordinate set*** $\{\theta_1, \theta_2\}$.



Find:

Using θ_1 and θ_2 as the single ***generalized coordinates***, find the ***differential equations of motion*** of the system.

Solution:

Using θ_1 and θ_2 as the two ***generalized coordinates***, the ***equations of motion*** of the system can be found using d'Alembert's principle**.**

$$\sum_{i=1}^{2}\left(m_i\,{}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial\dot{\theta}_k}\right)+\sum_{i=1}^{2}\left[\left(\underset{\sim}{I}_{G_i}\cdot{}^R\underset{\sim}{\alpha}_{L_i}\right)+\left({}^R\underset{\sim}{\omega}_{L_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_i}}{\partial\dot{\theta}_k}=F_{\theta_k}\qquad(k=1,2)$$

Kinematics:

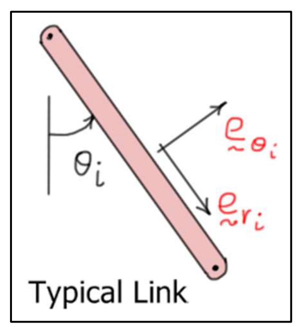
The angular velocities and partial angular velocities of the links can be written as

$${}^R\underset{\sim}{\omega}_{L_i}=\dot{\theta}_i\,\underset{\sim}{k}\quad(i=1,2)\qquad\qquad\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_i}}{\partial\dot{\theta}_j}=\begin{cases}0 & (i\neq j)\\ \underset{\sim}{k} & (i=j)\end{cases}\quad(i,j=1,2)$$

The velocities and partial velocities of the mass centers of the two links can be written as

$${}^R\underset{\sim}{v}_{G_1}={}^R\underset{\sim}{v}_{G_1/O}=\tfrac{1}{2}\ell\dot{\theta}_1\,\underset{\sim}{e}_{\theta_1}\qquad {}^R\underset{\sim}{v}_{G_2}={}^R\underset{\sim}{v}_{A/O}+{}^R\underset{\sim}{v}_{G_2/A}=\ell\dot{\theta}_1\,\underset{\sim}{e}_{\theta_1}+\tfrac{1}{2}\ell\dot{\theta}_2\,\underset{\sim}{e}_{\theta_2}$$

$$\frac{\partial\,{}^R\underset{\sim}{v}_{G_1}}{\partial\dot{\theta}_1}=\tfrac{1}{2}\ell\,\underset{\sim}{e}_{\theta_1}\qquad\frac{\partial\,{}^R\underset{\sim}{v}_{G_1}}{\partial\dot{\theta}_2}=\underset{\sim}{0}\qquad\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial\dot{\theta}_1}=\ell\,\underset{\sim}{e}_{\theta_1}\qquad\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial\dot{\theta}_2}=\tfrac{1}{2}\ell\,\underset{\sim}{e}_{\theta_2}$$



Finally, the accelerations of the mass centers of the two links can be written as

$${}^R\underset{\sim}{a}_{G_1}={}^R\underset{\sim}{a}_{G_1/O}=\tfrac{1}{2}\ell\ddot{\theta}_1\,\underset{\sim}{e}_{\theta_1}-\tfrac{1}{2}\ell\dot{\theta}_1^2\,\underset{\sim}{e}_{r_1}$$

$${}^R\underset{\sim}{a}_{G_2}={}^R\underset{\sim}{a}_{A/O}+{}^R\underset{\sim}{a}_{G_2/A}=\ell\ddot{\theta}_1\,\underset{\sim}{e}_{\theta_1}-\ell\dot{\theta}_1^2\,\underset{\sim}{e}_{r_1}+\tfrac{1}{2}\ell\ddot{\theta}_2\,\underset{\sim}{e}_{\theta_2}-\tfrac{1}{2}\ell\dot{\theta}_2^2\,\underset{\sim}{e}_{r_2}$$

Generalized Forces:

The active forces and torques in the system are the ***weight forces*** and the ***driving torques*** M_1 and M_2. Note that the torque M_1 is applied to the first link by the ground, and torque M_2 is applied to the second link by the first link. So, the first link experiences a torque of $M_1 - M_2$ while the second link experiences a torque of M_2.

The ***generalized forces*** can be calculated as follows.

$$F_{\theta_1} = \left(-mg\, \underset{\sim}{j} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \dot{\theta}_1} \right) + \left(-mg\, \underset{\sim}{j} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \dot{\theta}_1} \right) + \left((M_1 - M_2)\underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(M_2\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_1} \right)$$

$$= \left(-mg\, \underset{\sim}{j} \cdot \tfrac{1}{2}\ell\, \underset{\sim}{e}_{\theta_1} \right) + \left(-mg\, \underset{\sim}{j} \cdot \ell\, \underset{\sim}{e}_{\theta_1} \right) + M_1 - M_2$$

$$= -\tfrac{3}{2} mg\ell\, \underset{\sim}{j} \cdot \left(C_{\theta_1}\, \underset{\sim}{i} + S_{\theta_1} \underset{\sim}{j} \right) + M_1 - M_2$$

$$\Rightarrow \boxed{F_{\theta_1} = M_1 - M_2 - \tfrac{3}{2} mg\ell\, S_{\theta_1}}$$

$$F_{\theta_2} = \left(-mg\, \underset{\sim}{j} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \dot{\theta}_2} \right) + \left(-mg\, \underset{\sim}{j} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \dot{\theta}_2} \right) + \left((M_1 - M_2)\underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(M_2\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_2} \right)$$

$$= \left(-mg\, \underset{\sim}{j} \cdot \underset{\sim}{0} \right) + \left(-mg\, \underset{\sim}{j} \cdot \tfrac{1}{2}\ell\, \underset{\sim}{e}_{\theta_2} \right) + M_2$$

$$= -\tfrac{1}{2} mg\ell\, \underset{\sim}{j} \cdot \left(C_{\theta_2}\, \underset{\sim}{i} + S_{\theta_2}\, \underset{\sim}{j} \right) + M_2$$

$$\Rightarrow \boxed{F_{\theta_2} = M_2 - \tfrac{1}{2} mg\ell S_{\theta_2}}$$

Terms on left side of d'Alembert's principle can be written as

$$\boxed{m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \dot{\theta}_1} = m\left(\tfrac{1}{2}\ell\ddot{\theta}_1\, \underset{\sim}{e}_{\theta_1} - \tfrac{1}{2}\ell\dot{\theta}_1^2\, \underset{\sim}{e}_{r_1} \right) \cdot \left(\tfrac{1}{2}\ell\, \underset{\sim}{e}_{\theta_1} \right) = \tfrac{1}{4} m\ell^2 \ddot{\theta}_1} \quad \boxed{m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \dot{\theta}_2} = m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \underset{\sim}{0} = 0}$$

$$m_2\, {}^R\underset{\sim}{a}_{G_2} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \dot{\theta}_1} = m\left(\ell\ddot{\theta}_1 \underset{\sim}{e}_{\theta_1} - \ell\dot{\theta}_1^2\, \underset{\sim}{e}_{r_1} + \tfrac{1}{2}\ell\ddot{\theta}_2\, \underset{\sim}{e}_{\theta_2} - \tfrac{1}{2}\ell\dot{\theta}_2^2\, \underset{\sim}{e}_{r_2} \right) \cdot \ell\, \underset{\sim}{e}_{\theta_1}$$

$$= m\ell^2\ddot{\theta}_1 + \tfrac{1}{2} m\ell^2\ddot{\theta}_2 \left(\underset{\sim}{e}_{\theta_2} \cdot \underset{\sim}{e}_{\theta_1} \right) - \tfrac{1}{2} m\ell^2\dot{\theta}_2^2 \left(\underset{\sim}{e}_{r_2} \cdot \underset{\sim}{e}_{\theta_1} \right)$$

$$= m\ell^2\ddot{\theta}_1 + \tfrac{1}{2} m\ell^2\ddot{\theta}_2\, \underset{\sim}{e}_{\theta_1} \cdot \left(-S_{2-1}\, \underset{\sim}{e}_{r_1} + C_{2-1}\, \underset{\sim}{e}_{\theta_1} \right) - \tfrac{1}{2} m\ell^2\dot{\theta}_2^2\, \underset{\sim}{e}_{\theta_1} \cdot \left(C_{2-1}\, \underset{\sim}{e}_{r_1} + S_{2-1}\, \underset{\sim}{e}_{\theta_1} \right)$$

$$\Rightarrow \boxed{m_2\, {}^R\underset{\sim}{a}_{G_2} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \dot{\theta}_1} = m\ell^2\ddot{\theta}_1 + \tfrac{1}{2} m\ell^2 C_{2-1}\ddot{\theta}_2 - \tfrac{1}{2} m\ell^2 S_{2-1}\dot{\theta}_2^2}$$

$$m_2\, {}^R\underset{\sim}{a}_{G_2} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \dot{\theta}_2} = m\left(\ell\ddot{\theta}_1 \underset{\sim}{e}_{\theta_1} - \ell\dot{\theta}_1^2\, \underset{\sim}{e}_{r_1} + \tfrac{1}{2}\ell\ddot{\theta}_2\, \underset{\sim}{e}_{\theta_2} - \tfrac{1}{2}\ell\dot{\theta}_2^2\, \underset{\sim}{e}_{r_2} \right) \cdot \tfrac{1}{2}\ell\, \underset{\sim}{e}_{\theta_2}$$

$$= \tfrac{1}{2} m\ell^2\ddot{\theta}_1\, \underset{\sim}{e}_{\theta_1} \cdot \left(-S_{2-1}\, \underset{\sim}{e}_{r_1} + C_{2-1}\, \underset{\sim}{e}_{\theta_1} \right) - \tfrac{1}{2} m\ell^2\dot{\theta}_1^2\, \underset{\sim}{e}_{r_1} \cdot \left(-S_{2-1}\, \underset{\sim}{e}_{r_1} + C_{2-1}\, \underset{\sim}{e}_{\theta_1} \right) + \tfrac{1}{4} m\ell^2\ddot{\theta}_2$$

$$\Rightarrow \boxed{m_2\, {}^R\underset{\sim}{a}_{G_2} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \dot{\theta}_2} = \tfrac{1}{2} m\ell^2 C_{2-1}\ddot{\theta}_1 + \tfrac{1}{2} m\ell^2 S_{2-1}\dot{\theta}_1^2 + \tfrac{1}{4} m\ell^2\ddot{\theta}_2}$$

$$\boxed{\left(\underset{\sim}{I}_{G_1} \cdot {}^R\underset{\sim}{\alpha}_{L_1} \right) \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \dot{\theta}_1} = \left(\tfrac{1}{12} m\ell^2\ddot{\theta}_1\, \underset{\sim}{k} \right) \cdot \underset{\sim}{k} = \tfrac{1}{12} m\ell^2\ddot{\theta}_1} \quad \boxed{\left(\underset{\sim}{I}_{G_1} \cdot {}^R\underset{\sim}{\alpha}_{L_1} \right) \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \dot{\theta}_2} = \left(\tfrac{1}{12} m\ell^2\ddot{\theta}_1\, \underset{\sim}{k} \right) \cdot \underset{\sim}{0} = 0}$$

$$\left(\underset{\sim}{I}_{G_2}\cdot {}^R\underset{\sim}{\alpha}_{L_2}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_2}}{\partial\dot{\theta}_1}=\left(\tfrac{1}{12}m\,\ell^2\ddot{\theta}_2\,\underset{\sim}{k}\right)\cdot\underset{\sim}{0}=0$$

$$\left(\underset{\sim}{I}_{G_2}\cdot {}^R\underset{\sim}{\alpha}_{L_2}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_2}}{\partial\dot{\theta}_2}=\left(\tfrac{1}{12}m\,\ell^2\ddot{\theta}_2\,\underset{\sim}{k}\right)\cdot\underset{\sim}{k}=\tfrac{1}{12}m\,\ell^2\ddot{\theta}_2$$

Recall finally for two dimensional problems that $\boxed{{}^R\underset{\sim}{\omega}_{L_i}\times\underset{\sim}{H}_{G_i}=\underset{\sim}{0}}$.

d'Alembert's Equations of Motion:

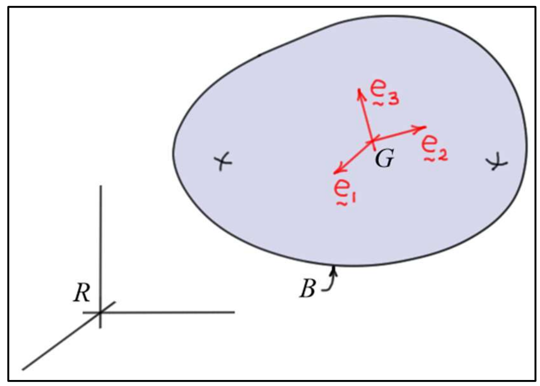
Substituting the above results into d'Alembert's principle gives

$$\tfrac{4}{3}m\,\ell^2\,\ddot{\theta}_1+\tfrac{1}{2}m\,\ell^2C_{2-1}\ddot{\theta}_2-\tfrac{1}{2}m\,\ell^2S_{2-1}\,\dot{\theta}_2^2+\tfrac{3}{2}mg\ell\,S_{\theta_1}=M_1-M_2$$

$$\tfrac{1}{2}m\,\ell^2C_{2-1}\ddot{\theta}_1+\tfrac{1}{3}m\,\ell^2\ddot{\theta}_2+\tfrac{1}{2}m\,\ell^2S_{2-1}\dot{\theta}_1^2+\tfrac{1}{2}mg\ell S_{\theta_2}=M_2$$

These are two ***second-order, ordinary differential equations*** for the generalized coordinates θ_1 and θ_2, and they are ***identical*** to those found using Lagrange's equations in Unit 4 of this volume.

Example 2: Unconstrained Motion of a Rigid Body



The rigid body B with mass center G is moving freely in the inertial frame R. The unit vectors of the body frame $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ represent the principal directions associated with G and are related to those in the inertial frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ by the transformation matrix $[R]$. The velocity of the mass center G is given in the body and inertial frames as

$${}^R\underset{\sim}{v}_G=u\,\underset{\sim}{e}_1+v\,\underset{\sim}{e}_2+w\,\underset{\sim}{e}_3 \qquad {}^R\underset{\sim}{v}_G=\dot{X}\,\underset{\sim}{N}_1+\dot{Y}\,\underset{\sim}{N}_2+\dot{Z}\,\underset{\sim}{N}_3$$

These velocity components are related using the transformation matrix $[R]$ as follows.

$$\begin{Bmatrix}u\\v\\w\end{Bmatrix}=[R]\begin{Bmatrix}\dot{X}\\\dot{Y}\\\dot{Z}\end{Bmatrix}$$

The angular velocity of the body is written in body-fixed components as

$${}^R\underset{\sim}{\omega}_B=\omega_1\,\underset{\sim}{e}_1+\omega_2\,\underset{\sim}{e}_2+\omega_3\,\underset{\sim}{e}_3$$

Find:

The EOM of the rigid body using Kane's equations. Use ***Euler parameters*** to describe the orientation of the body and define the ***six generalized speeds*** as $[u_1,u_2,u_3,u_4,u_5,u_6]=[u,v,w,\omega_1,\omega_2,\omega_3]$.

Solution:

Using the generalized speeds defined above, Kane' equations of motion for the rigid body can be written as

$$\boxed{\left(m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_k}\right)+\left[\left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)+\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_k}=F_{u_k}}\quad (k=1,\ldots,6)$$

Partial Velocities:

The partial angular velocities of the body and the partial velocities associated with the mass center of the body can be written as

$$\boxed{\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_k}=\underset{\sim}{e}_k}\quad (k=1,2,3)\qquad \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_k}=\underset{\sim}{0}}\quad (k=4,5,6)$$

$$\boxed{\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_k}=\underset{\sim}{0}}\quad (k=1,2,3)\qquad \boxed{\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_k}=\underset{\sim}{e}_{k-3}}\quad (k=4,5,6)$$

Mass Center Acceleration:

The acceleration of the mass center of the body can be written as

$${}^R\underset{\sim}{a}_G=\frac{{}^Rd\,{}^R\underset{\sim}{v}_G}{dt}=\frac{{}^Bd\,{}^R\underset{\sim}{v}_G}{dt}+\left({}^R\underset{\sim}{\omega}_B\times{}^R\underset{\sim}{v}_G\right)=\left(\dot{u}\,\underset{\sim}{e}_1+\dot{v}\,\underset{\sim}{e}_2+\dot{w}\,\underset{\sim}{e}_3\right)+\begin{vmatrix}\underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3\\ \omega_1 & \omega_2 & \omega_3\\ u & v & w\end{vmatrix}$$

$$\Rightarrow\boxed{{}^R\underset{\sim}{a}_G=\left(\dot{u}+\omega_2 w-\omega_3 v\right)\underset{\sim}{e}_1+\left(\dot{v}+\omega_3 u-\omega_1 w\right)\underset{\sim}{e}_2+\left(\dot{w}+\omega_1 v-\omega_2 u\right)\underset{\sim}{e}_3}$$

Terms on the Left Side of Kane's equations:

Using the results from above, the terms on the left side of Kane's equations can be written as

$$\boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_i}=m\left[\left(\dot{u}+\omega_2 w-\omega_3 v\right)\underset{\sim}{e}_1+\left(\dot{v}+\omega_3 u-\omega_1 w\right)\underset{\sim}{e}_2+\left(\dot{w}+\omega_1 v-\omega_2 u\right)\underset{\sim}{e}_3\right]\cdot\underset{\sim}{e}_i}\quad (k=1,2,3)$$

$$\Rightarrow\boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_1}=m\left(\dot{u}+\omega_2 w-\omega_3 v\right)}\ \boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_2}=m\left(\dot{v}+\omega_3 u-\omega_1 w\right)}$$

$$\boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_3}=m\left(\dot{w}+\omega_1 v-\omega_2 u\right)}$$

$$\boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_i}=0}\quad (k=4,5,6)$$

$$\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\ \rightarrow\begin{bmatrix}I_{11}^G & 0 & 0\\ 0 & I_{22}^G & 0\\ 0 & 0 & I_{33}^G\end{bmatrix}\begin{Bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{Bmatrix}\rightarrow\boxed{\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B=I_{11}^G\,\dot{\omega}_1\,\underset{\sim}{e}_1+I_{22}^G\,\dot{\omega}_2\,\underset{\sim}{e}_2+I_{33}^G\,\dot{\omega}_3\,\underset{\sim}{e}_3}$$

$$\boxed{\left(\underset{\sim}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_k}=0} \quad (k=1,2,3) \qquad \boxed{\left(\underset{\sim}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_{k+3}}=I^G_{kk}\,\dot{\omega}_k} \quad (k=1,2,3)$$

$$\begin{aligned}{}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G &= \begin{vmatrix}\underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3\\ \omega_1 & \omega_2 & \omega_3\\ I^G_{11}\omega_1 & I^G_{22}\omega_2 & I^G_{33}\omega_3\end{vmatrix}\\ &=\left(I^G_{33}-I^G_{22}\right)\omega_2\omega_3\,\underset{\sim}{e}_1+\left(I^G_{11}-I^G_{33}\right)\omega_1\omega_3\,\underset{\sim}{e}_2+\left(I^G_{22}-I^G_{11}\right)\omega_1\omega_2\,\underset{\sim}{e}_3\end{aligned}$$

$$\boxed{\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_k}=0} \quad (k=1,2,3)$$

$$\Rightarrow \boxed{\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_4}=\left(I^G_{33}-I^G_{22}\right)\omega_2\omega_3} \quad \boxed{\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_5}=\left(I^G_{11}-I^G_{33}\right)\omega_1\omega_3}$$

$$\boxed{\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial u_6}=\left(I^G_{22}-I^G_{11}\right)\omega_1\omega_2}$$

Kane’s Equations of Motion:

Substituting into Kane’s equations of motion gives

$$\boxed{\begin{aligned}m\left(\dot{u}+\omega_2 w-\omega_3 v\right)&=F_u\\ m\left(\dot{v}+\omega_3 u-\omega_1 w\right)&=F_v\\ m\left(\dot{w}+\omega_1 v-\omega_2 u\right)&=F_w\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned}I^G_{11}\,\dot{\omega}_1+\left(I^G_{33}-I^G_{22}\right)\omega_2\omega_3&=F_{\omega_1}\\ I^G_{22}\,\dot{\omega}_2+\left(I^G_{11}-I^G_{33}\right)\omega_1\omega_3&=F_{\omega_2}\\ I^G_{33}\,\dot{\omega}_3+\left(I^G_{22}-I^G_{11}\right)\omega_1\omega_2&=F_{\omega_3}\end{aligned}}$$

Kinematical Differential Equations:

The kinematical differential equations that relate the angular velocity components to the Euler parameters and the body-fixed velocity components to the inertial velocity components can be written as

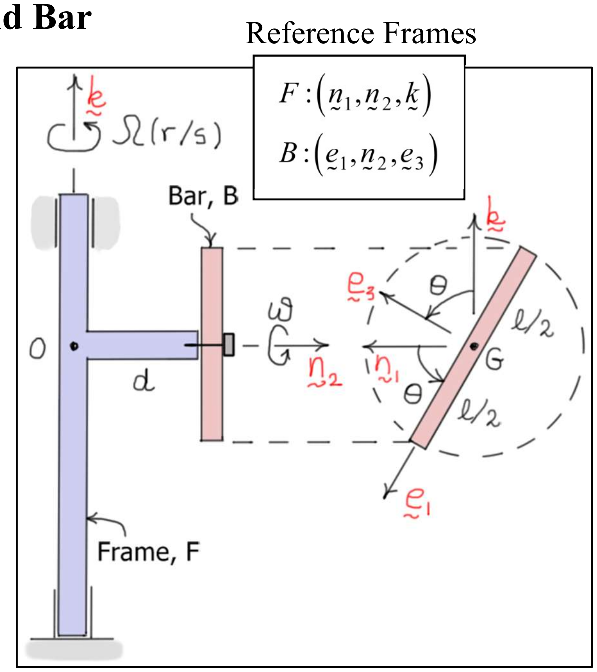
$$\boxed{\begin{Bmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\\ \dot{\varepsilon}_3\\ \dot{\varepsilon}_4\end{Bmatrix}=\tfrac{1}{2}\left[E'\right]^T\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ 0\end{Bmatrix}=\frac{1}{2}\begin{bmatrix}\varepsilon_4 & \varepsilon_3 & -\varepsilon_2 & -\varepsilon_1\\ -\varepsilon_3 & \varepsilon_4 & \varepsilon_1 & -\varepsilon_2\\ \varepsilon_2 & -\varepsilon_1 & \varepsilon_4 & -\varepsilon_3\\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4\end{bmatrix}^T\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ 0\end{Bmatrix}} \quad \boxed{\begin{Bmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\end{Bmatrix}=\left[R\right]^T\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}}$$

Recall here that the last of the four Euler parameter differential equations is simply the derivative of the constraint that relates the four parameters, that is, $\dfrac{d}{dt}\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)=0$.

Together, ***Kane’s equations*** and the ***kinematical differential equations*** represent a set of ***thirteen first-order, ordinary differential equations*** in the ***thirteen unknowns*** $\left\{X,Y,Z,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,u,v,w,\omega_1,\omega_2,\omega_3\right\}$.

Note: The form of d'Alembert's principle given above ***could not be used*** for this problem because the four Euler parameters are ***not independent***. When using d'Alembert's principle with a set of ***dependent generalized coordinates***, Lagrange multipliers are used. See Unit 6 of this volume for applications using Lagrange multipliers.

Example 3: Three-Dimensional Rotating Frame and Bar



The system shown has ***two degrees of freedom*** and consists of two bodies, the frame *F* and the bar *B*. As *F* rotates about the fixed vertical direction, *B* rotates relative to the horizontal arm of *F*. The orientation of *F* is given by the angle ϕ $(\dot{\phi}=\Omega)$, and the orientation of *B* is given by the angle θ $(\dot{\theta}=\omega)$. The bar has mass *m* and length ℓ. The frame is assumed to be light. The motor torque $M_\phi(t)$ is applied to *F* by the ground, and the motor torque $M_\theta(t)$ is applied to *B* by *F*.

Find:

The equations of motion of the system associated with the angles ϕ and θ.

Solution:

Using the angles ϕ and θ as a set of ***independent generalized coordinates***, the equations of motion of the bar can be found using d'Alembert's principle.

$$\left(m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial\dot{\phi}}\right)+\left[\left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)+\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\phi}}=F_\phi$$

$$\left(m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial\dot{\theta}}\right)+\left[\left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)+\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}}=F_\theta$$

Kinematics:

The ***angular velocity***, ***partial angular velocities*** and ***angular momentum*** of the bar can be written as

$$\boxed{{}^R\underset{\sim}{\omega}_B={}^R\underset{\sim}{\omega}_F+{}^F\underset{\sim}{\omega}_B=\dot{\phi}\,\underset{\sim}{k}+\dot{\theta}\,\underset{\sim}{n}_2}\quad\Rightarrow\quad\boxed{\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\phi}}=\underset{\sim}{k}=-S_\theta\,\underset{\sim}{e}_1+C_\theta\,\underset{\sim}{e}_3}\quad\text{and}\quad\boxed{\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}}=\underset{\sim}{n}_2}$$

$$\underset{\sim}{H}_G=\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\omega}_B\quad\to\tfrac{1}{12}m\ell^2\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}-\dot{\phi}S_\theta\\\dot{\theta}\\\dot{\phi}C_\theta\end{Bmatrix}\quad\to\boxed{\underset{\sim}{H}_G=\tfrac{1}{12}m\ell^2\left(\dot{\theta}\,\underset{\sim}{n}_2+\dot{\phi}\,C_\theta\,\underset{\sim}{e}_3\right)}$$

The angular acceleration of the bar is found by differentiating the angular velocity.

$$ {}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left(\dot{\phi}\,\underset{\sim}{k} + \dot{\theta}\,\underset{\sim}{n}_2\right) = \ddot{\phi}\,\underset{\sim}{k} + \ddot{\theta}\,\underset{\sim}{n}_2 - \dot{\theta}\dot{\phi}\,\underset{\sim}{n}_1 = \ddot{\phi}\left(-S_\theta\,\underset{\sim}{e}_1 + C_\theta\,\underset{\sim}{e}_3\right) + \ddot{\theta}\,\underset{\sim}{n}_2 - \dot{\theta}\dot{\phi}\left(C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{e}_3\right) $$

$$ \Rightarrow \boxed{{}^R\underset{\sim}{\alpha}_B = -\left(S_\theta\ddot{\phi} + C_\theta\dot{\theta}\dot{\phi}\right)\underset{\sim}{e}_1 + \left(\ddot{\theta}\right)\underset{\sim}{n}_2 + \left(C_\theta\ddot{\phi} - S_\theta\dot{\theta}\dot{\phi}\right)\underset{\sim}{e}_3} $$

Finally, the velocity of the mass center G and the corresponding partial velocities can be written as

$$ \boxed{{}^R\underset{\sim}{v}_G = -d\dot{\phi}\,\underset{\sim}{n}_1} \qquad \Rightarrow \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial\dot{\phi}} = -d\,\underset{\sim}{n}_1} \quad \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial\dot{\theta}} = \underset{\sim}{0}} $$

Generalized Forces:

The only active forces and torques in the system are the ***motor torques***. The frame F is subjected to the motor torque $M_\phi(t)$ and to the reaction torque $-M_\theta(t)$, and bar B is subjected to motor torque $M_\theta(t)$. The ***generalized forces*** associated with these torques are

$$ \boxed{F_\theta = \left(\left(M_\phi\,\underset{\sim}{k} - M_\theta\,\underset{\sim}{n}_2\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial\dot{\theta}}\right) + \left(M_\theta\,\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}}\right) = M_\theta(t)} $$

$$ \boxed{F_\phi = \left(\left(M_\phi\,\underset{\sim}{k} - M_\theta\,\underset{\sim}{n}_2\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial\dot{\phi}}\right) + \left(M_\theta\,\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\phi}}\right) = M_\phi(t)} $$

Terms on left side of d'Alembert's principle:

The terms on the left side of d'Alembert's principle can be written as

$$ \boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial\dot{\phi}} = m\left(-d\ddot{\phi}\,\underset{\sim}{n}_1 - d\dot{\phi}^2\,\underset{\sim}{n}_2\right)\cdot\left(-d\,\underset{\sim}{n}_1\right) = md^2\ddot{\phi}} \quad \boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial\dot{\theta}} = m\,{}^R\underset{\sim}{a}_G\cdot\underset{\sim}{0} = 0} $$

$$ \underset{\sim}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B \quad \rightarrow \tfrac{1}{12}m\ell^2\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}-\left(S_\theta\ddot{\phi} + C_\theta\dot{\theta}\dot{\phi}\right)\\ \ddot{\theta}\\ \left(C_\theta\ddot{\phi} - S_\theta\dot{\theta}\dot{\phi}\right)\end{Bmatrix} \rightarrow \boxed{\underset{\sim}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B = \tfrac{1}{12}m\ell^2\left[\ddot{\theta}\,\underset{\sim}{n}_2 + \left(C_\theta\ddot{\phi} - S_\theta\dot{\theta}\dot{\phi}\right)\underset{\sim}{e}_3\right]} $$

$$ \boxed{\left(\underset{\sim}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\phi}} = \tfrac{1}{12}m\ell^2\left[\ddot{\theta}\,\underset{\sim}{n}_2 + \left(C_\theta\ddot{\phi} - S_\theta\dot{\theta}\dot{\phi}\right)\underset{\sim}{e}_3\right]\cdot\left(-S_\theta\,\underset{\sim}{e}_1 + C_\theta\,\underset{\sim}{e}_3\right) = \tfrac{1}{12}m\ell^2\left(C_\theta\ddot{\phi} - S_\theta\dot{\theta}\dot{\phi}\right)C_\theta} $$

$$ \boxed{\left(\underset{\sim}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}} = \tfrac{1}{12}m\ell^2\left[\ddot{\theta}\,\underset{\sim}{n}_2 + \left(C_\theta\ddot{\phi} - S_\theta\dot{\theta}\dot{\phi}\right)\underset{\sim}{e}_3\right]\cdot\,\underset{\sim}{n}_2 = \tfrac{1}{12}m\ell^2\ddot{\theta}} $$

$$ \boxed{\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\phi}} = \begin{vmatrix}-S_\theta & 0 & C_\theta\\ -\dot{\phi}S_\theta & \dot{\theta} & \dot{\phi}C_\theta\\ 0 & \tfrac{1}{12}m\ell^2\dot{\theta} & \tfrac{1}{12}m\ell^2\dot{\phi}C_\theta\end{vmatrix} = -S_\theta(0) + C_\theta\left(-\tfrac{1}{12}m\ell^2 S_\theta\dot{\theta}\dot{\phi}\right) = -\tfrac{1}{12}m\ell^2 S_\theta C_\theta\dot{\theta}\dot{\phi}} $$

$$\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}} = \begin{vmatrix} 0 & 1 & 0 \\ -\dot{\phi}S_\theta & \dot{\theta} & \dot{\phi}C_\theta \\ 0 & \frac{1}{12}m\ell^2\dot{\theta} & \frac{1}{12}m\ell^2\dot{\phi}C_\theta \end{vmatrix} = \tfrac{1}{12}m\ell^2\, S_\theta C_\theta \dot{\phi}^2$$

d'Alembert's Equations of Motion:

Substituting these results into d'Alembert's principle gives

$$\left(\tfrac{1}{12}m\ell^2 C_\theta^2 + md^2\right)\ddot{\phi} - \tfrac{1}{6}m\ell^2 S_\theta C_\theta \dot{\theta}\dot{\phi} = M_\phi$$

$$\tfrac{1}{12}m\ell^2\ddot{\theta} + \tfrac{1}{12}m\ell^2\, S_\theta C_\theta \dot{\phi}^2 = M_\theta$$

These are two ***second-order, ordinary differential equations*** for the generalized coordinates ϕ and θ, and they are ***identical*** to those found using Lagrange's equations in Unit 4 of this volume.

Example 4: Three-Dimensional Rotating Frame and Bar

Find:

Solve the problem of Example 3 using Kane's equations. Use $\{v_G, \omega_2\}$ as the independent set of generalized speeds, where $v_G = d\,\dot{\phi}$ and $\omega_2 = \dot{\theta}$.

Solution:

Using the generalized speeds given above, Kane's equations can be written as follows.

$$\left(m\,{}^R\underset{\sim}{a}_G \cdot \frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial v_G}\right) + \left[\left(\underset{\approx}{I}_G\cdot {}^R\underset{\sim}{\alpha}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial v_G} = F_{v_G}$$

$$\left(m\,{}^R\underset{\sim}{a}_G \cdot \frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial \omega_2}\right) + \left[\left(\underset{\approx}{I}_G\cdot {}^R\underset{\sim}{\alpha}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_2} = F_{\omega_2}$$

Kinematics:

The ***angular velocity***, ***partial angular velocities*** and ***angular momentum*** of the bar can be written as

$${}^R\underset{\sim}{\omega}_B = {}^R\underset{\sim}{\omega}_F + {}^F\underset{\sim}{\omega}_B = \tfrac{v_G}{d}\,\underset{\sim}{k} + \omega_2\,\underset{\sim}{n}_2 \quad\Rightarrow\quad \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial v_G} = \tfrac{1}{d}\underset{\sim}{k} = \tfrac{1}{d}\left(-S_\theta\,\underset{\sim}{e}_1 + C_\theta\,\underset{\sim}{e}_3\right) \quad\text{and}\quad \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_2} = \underset{\sim}{n}_2$$

$$\underset{\sim}{H}_G = \underset{\approx}{I}_G\cdot {}^R\underset{\sim}{\omega}_B \quad \to \tfrac{1}{12}m\ell^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -\tfrac{1}{d}v_G\,S_\theta \\ \omega_2 \\ \tfrac{1}{d}v_G\,C_\theta \end{Bmatrix} \quad\to\quad \underset{\sim}{H}_G = \tfrac{1}{12}m\ell^2\left(\omega_2\,\underset{\sim}{n}_2 + \tfrac{1}{d}v_G\,C_\theta\,\underset{\sim}{e}_3\right)$$

The angular acceleration of the bar is

$$\begin{aligned} {}^R\underset{\sim}{\alpha}_B &= \frac{{}^R d}{dt}\left(\tfrac{1}{d}v_G\,\underset{\sim}{k} + \omega_2\,\underset{\sim}{n}_2\right) = \tfrac{1}{d}\dot{v}_G\,\underset{\sim}{k} + \dot{\omega}_2\,\underset{\sim}{n}_2 + \omega_2\left(\tfrac{1}{d}v_G\,\underset{\sim}{k}\times\underset{\sim}{n}_2\right) = \tfrac{1}{d}\dot{v}_G\,\underset{\sim}{k} + \dot{\omega}_2\,\underset{\sim}{n}_2 - \omega_2\tfrac{1}{d}v_G\,\underset{\sim}{n}_1 \\ &= \tfrac{1}{d}\dot{v}_G\left(-S_\theta\,\underset{\sim}{e}_1 + C_\theta\,\underset{\sim}{e}_3\right) + \dot{\omega}_2\,\underset{\sim}{n}_2 - \tfrac{1}{d}v_G\omega_2\left(C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{e}_3\right) \end{aligned}$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{\alpha}_B = -\tfrac{1}{d}\left(\dot{v}_G S_\theta + v_G\omega_2 C_\theta\right)\underset{\sim}{e}_1 + \left(\dot{\omega}_2\right)\underset{\sim}{n}_2 + \tfrac{1}{d}\left(\dot{v}_G C_\theta - v_G\omega_2 S_\theta\right)\underset{\sim}{e}_3}$$

Finally, the velocity of the mass center G and the corresponding partial velocities are

$$\boxed{{}^R\underset{\sim}{v}_G = -v_G\,\underset{\sim}{n}_1} \qquad \Rightarrow \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial v_G} = -\underset{\sim}{n}_1} \quad \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial \omega_2} = \underset{\sim}{0}}$$

Generalized Forces:

The only active forces or torques in the system are the ***motor torques***. The frame F is subjected to the motor torque $M_\phi(t)$ and to the reaction torque $-M_\theta(t)$, and bar B is subjected to motor torque $M_\theta(t)$. The ***generalized forces*** associated with these torques are

$$\boxed{F_{v_G} = \left(\left(M_\phi\,\underset{\sim}{k} - M_\theta\,\underset{\sim}{n}_2\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial v_G}\right) + \left(M_\theta\,\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial v_G}\right) = \tfrac{1}{d}M_\phi(t)}$$

$$\boxed{F_{\omega_2} = \left(\left(M_\phi\,\underset{\sim}{k} - M_\theta\,\underset{\sim}{n}_2\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial \omega_2}\right) + \left(M_\theta\,\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial \omega_2}\right) = M_\theta(t)}$$

Terms on left side of Kane's Equations:

The terms on the left side of Kane's equations can be written as

$$\boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial v_G} = m\left(-\dot{v}_G\,\underset{\sim}{n}_1 - \tfrac{1}{d}v_G^2\,\underset{\sim}{n}_2\right)\cdot\left(-\underset{\sim}{n}_1\right) = m\,\dot{v}_G} \quad \boxed{m\,{}^R\underset{\sim}{a}_G\cdot\frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial \omega_2} = m\,{}^R\underset{\sim}{a}_G\cdot\underset{\sim}{0} = 0}$$

$$\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B \quad\to\quad \tfrac{1}{12}m\ell^2\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}-\tfrac{1}{d}\left(\dot{v}_G S_\theta + v_G\omega_2 C_\theta\right)\\ \dot{\omega}_2\\ \tfrac{1}{d}\left(\dot{v}_G C_\theta - v_G\omega_2 S_\theta\right)\end{Bmatrix}$$

$$\Rightarrow \boxed{\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B = \tfrac{1}{12}m\ell^2\left[\dot{\omega}_2\,\underset{\sim}{n}_2 + \tfrac{1}{d}\left(\dot{v}_G C_\theta - v_G\omega_2 S_\theta\right)\underset{\sim}{e}_3\right]}$$

$$\boxed{\begin{aligned}\left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial v_G} &= \tfrac{1}{12}m\ell^2\left[\dot{\omega}_2\,\underset{\sim}{n}_2 + \tfrac{1}{d}\left(\dot{v}_G C_\theta - v_G\omega_2 S_\theta\right)\underset{\sim}{e}_3\right]\cdot\tfrac{1}{d}\left(-S_\theta\,\underset{\sim}{e}_1 + C_\theta\,\underset{\sim}{e}_3\right)\\ &= \tfrac{1}{12\,d^2}m\ell^2\left(\dot{v}_G C_\theta - v_G\omega_2 S_\theta\right)C_\theta\end{aligned}}$$

$$\boxed{\left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial \omega_2} = \tfrac{1}{12}m\ell^2\left[\dot{\omega}_2\,\underset{\sim}{n}_2 + \tfrac{1}{d}\left(\dot{v}_G C_\theta - v_G\omega_2 S_\theta\right)\underset{\sim}{e}_3\right]\cdot\,\underset{\sim}{n}_2 = \tfrac{1}{12}m\ell^2\dot{\omega}_2}$$

$$\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial v_G} = \begin{vmatrix} -\frac{1}{d}S_\theta & 0 & \frac{1}{d}C_\theta \\ -\frac{v_G}{d}S_\theta & \omega_2 & \frac{v_G}{d}C_\theta \\ 0 & \frac{1}{12}m\ell^2\omega_2 & \frac{1}{12d}m\ell^2 v_G C_\theta \end{vmatrix} = -\frac{1}{d}S_\theta(0) + \frac{1}{d}C_\theta\left(-\frac{1}{12}m\ell^2 S_\theta\omega_2\frac{v_G}{d}\right)$$

$$= -\frac{1}{12d^2}m\ell^2 S_\theta C_\theta \omega_2 v_G$$

$$\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_2} = \begin{vmatrix} 0 & 1 & 0 \\ -\frac{v_G}{d}S_\theta & \omega_2 & \frac{v_G}{d}C_\theta \\ 0 & \frac{1}{12}m\ell^2\omega_2 & \frac{1}{12d}m\ell^2 v_G C_\theta \end{vmatrix} = \frac{1}{12d^2}m\ell^2 S_\theta C_\theta v_G^2$$

Equations of Motion:

Substituting these results into Kane's equations gives

$$\boxed{\left(m\,d + \frac{1}{12\,d}m\ell^2 C_\theta^2\right)\dot{v}_G - \frac{1}{6\,d}m\ell^2 S_\theta C_\theta \omega_2 v_G = M_\phi(t)}$$

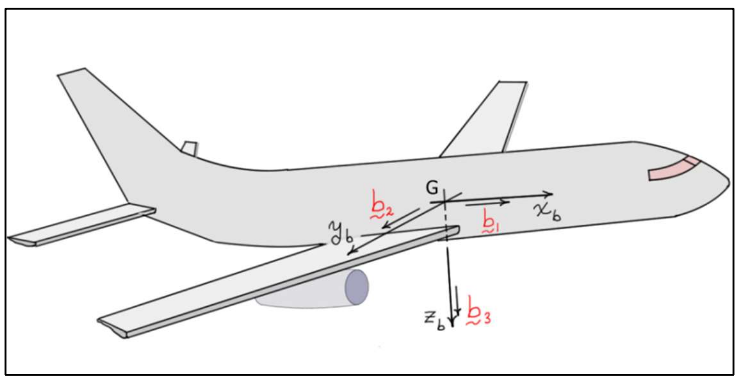
$$\boxed{\frac{1}{12}m\ell^2\dot{\omega}_2 + \frac{1}{12d^2}m\ell^2 S_\theta C_\theta v_G^2 = M_\theta}$$

These are two ***first-order, ordinary differential equations*** for the generalized speeds v_G and ω_2. To solve, these equations are ***supplemented*** with the ***two kinematical differential equations***.

$\boxed{\dot{\phi} = \frac{1}{d}v_G}$ and $\boxed{\dot{\theta} = \omega_2}$

Example 5: Aircraft with Two Engines

The aircraft shown has two engines, one on each wing. The orientation of the aircraft relative to a fixed reference frame R is defined by a 3-2-1 ***body-fixed*** rotation sequence (ψ,θ,ϕ). For the purposes of this example, the ***aircraft*** consists of ***three*** main components, the ***airframe*** A and the ***two engines*** E_1 and E_2.



The term ***airframe*** is used to refer to all the ***stationary components*** of the aircraft, and the term ***engine*** is used to refer to the ***rotating components*** of the engines. The points G_i $(i=1,2)$ are the mass centers of the two ***engines***, G_A is the mass center of the ***airframe***, and G is the mass center of the ***aircraft***.

The aircraft is ***symmetrical*** with respect to the $x_b z_b$ plane. The ***two engines*** are assumed to be ***identical*** and ***placed symmetrically*** on the airframe so the ***position vector*** of G_A the ***mass center*** of the ***airframe*** and the

position vectors of $G_i\ (i=1,2)$ the ***mass centers*** of the ***engines*** relative to G the ***mass center*** of the ***aircraft*** can be written as

$$\boxed{\underset{\sim}{r}_{G_A/G} = x_A\,\underset{\sim}{b}_1 + z_A\,\underset{\sim}{b}_3} \qquad \boxed{\underset{\sim}{r}_{G_1/G} = x_E\,\underset{\sim}{b}_1 + y_E\,\underset{\sim}{b}_2 + z_E\,\underset{\sim}{b}_3} \qquad \boxed{\underset{\sim}{r}_{G_2/G} = x_E\,\underset{\sim}{b}_1 - y_E\,\underset{\sim}{b}_2 + z_E\,\underset{\sim}{b}_3}$$

Furthermore, the engines (rotating components) are assumed to be ***solids of revolution*** aligned with the x_b axis (meaning they are ***rotationally symmetrical*** about that axis). Finally, the ***velocity*** of G the ***mass center*** of the ***aircraft*** is specified in the ***body*** and ***ground frames*** as

$$\boxed{{}^R\underset{\sim}{v}_G = u\,\underset{\sim}{b}_1 + v\,\underset{\sim}{b}_2 + w\,\underset{\sim}{b}_3} \qquad \boxed{{}^R\underset{\sim}{v}_G = \dot{X}\,\underset{\sim}{N}_1 + \dot{Y}\,\underset{\sim}{N}_2 + \dot{Z}\,\underset{\sim}{N}_3}$$

The angular velocity of the airframe is expressed in body-fixed components as

$$\boxed{{}^R\underset{\sim}{\omega}_A = \omega_1\,\underset{\sim}{b}_1 + \omega_2\,\underset{\sim}{b}_2 + \omega_3\,\underset{\sim}{b}_3 = \left(\dot{\phi} - \dot{\psi}\,S_\theta\right)\underset{\sim}{b}_1 + \left(\dot{\theta}\,C_\phi + \dot{\psi}\,C_\theta\,S_\phi\right)\underset{\sim}{b}_2 + \left(-\dot{\theta}\,S_\phi + \dot{\psi}\,C_\theta\,C_\phi\right)\underset{\sim}{b}_3}$$

Reference frame:

$R: \underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3$ (ground frame – fixed on the ground)

$A: \underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$ (fixed in the airframe)

Find: (assuming all external forces and torques and engine speeds are known)

Equations of motion of the system consisting of the engines and airframe

Use $\left\{u_1, u_2, u_3, u_4, u_5, u_6\right\} = \left\{u, v, w, \omega_1, \omega_2, \omega_3\right\}$ as the ***six independent generalized speeds***

Solution:

In the following solution, it is assumed that the external forces acting on the airframe are replaced by a single force $\underset{\sim}{F}^A_{\text{ext}}$ acting at G_A the mass center of the airframe and a corresponding torque $\underset{\sim}{T}^A_{\text{ext}}$. The external forces on the two engines are replaced by single forces $\underset{\sim}{F}^{E_i}_{\text{ext}}\ (i=1,2)$ acting at $G_i\ (i=1,2)$ the mass centers of the engines and corresponding torques $\underset{\sim}{T}^{E_i}_{\text{ext}}\ (i=1,2)$. These forces and torques are then replaced by a ***single resultant force*** $\underset{\sim}{F}_{\text{ext}}$ acting at G the mass center of the aircraft and a ***corresponding torque*** $\underset{\sim}{T}^G_{\text{ext}}$ where

$$\boxed{\underset{\sim}{F}_{\text{ext}} = \underset{\sim}{F}^A_{\text{ext}} + \sum_{i=1}^{2}\underset{\sim}{F}^{G_i}_{\text{ext}}} \qquad \boxed{\underset{\sim}{T}^G_{\text{ext}} = \underset{\sim}{T}^A_{\text{ext}} + \sum_{i=1}^{2}\underset{\sim}{T}^{G_i}_{\text{ext}} + \left(\underset{\sim}{r}_{G_A/G} \times \underset{\sim}{F}^A_{\text{ext}}\right) + \sum_{i=1}^{2}\left(\underset{\sim}{r}_{G_i/G} \times \underset{\sim}{F}^{G_i}_{\text{ext}}\right)}$$

See comments in Addendum 1 of Unit 4 regarding ***generalized forces*** when using ***equivalent force systems*** on a rigid body.

<u>Previous results</u>

In Volume I, Unit 5 the ***angular velocity*** and ***angular acceleration*** of the ***airframe*** were found to be (using a 3-2-1 body-fixed, orientation angle sequence)

$$\boxed{{}^R\underset{\sim}{\omega}_B = \left(\dot{\phi} - \dot{\psi}\,S_\theta\right)\underset{\sim}{b}_1 + \left(\dot{\theta}\,C_\phi + \dot{\psi}\,C_\theta\,S_\phi\right)\underset{\sim}{b}_2 + \left(-\dot{\theta}\,S_\phi + \dot{\psi}\,C_\theta\,C_\phi\right)\underset{\sim}{b}_3 \triangleq \omega_1\,\underset{\sim}{e}_1 + \omega_2\,\underset{\sim}{e}_2 + \omega_3\,\underset{\sim}{e}_3}$$

and

$$ {}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \frac{{}^B d}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \dot{\omega}_1\,\underset{\sim}{b}_1 + \dot{\omega}_2\,\underset{\sim}{b}_2 + \dot{\omega}_3\,\underset{\sim}{b}_3 $$

In Unit 1 of this volume, the ***inertia matrix*** and ***angular momentum*** of the ***airframe*** were written as follows. Note the ***zero values*** for some of the ***products of inertia*** due to the ***assumed symmetry*** of the airframe.

$$ \left[I_{G_A}\right]_A = \begin{bmatrix} I_{x_b x_b}^{G_A} & 0 & -I_{x_b z_b}^{G_A} \\ 0 & I_{y_b y_b}^{G_A} & 0 \\ -I_{x_b z_b}^{G_A} & 0 & I_{z_b z_b}^{G_A} \end{bmatrix} $$

$$ \underset{\sim}{H}_{G_A} = \left(I_{x_b x_b}^{G_A}\omega_1 - I_{x_b z_b}^{G_A}\omega_3\right)\underset{\sim}{b}_1 + \left(I_{y_b y_b}^{G_A}\omega_2\right)\underset{\sim}{b}_2 + \left(-I_{x_b z_b}^{G_A}\omega_1 + I_{z_b z_b}^{G_A}\omega_3\right)\underset{\sim}{b}_3 $$

The ***angular velocities***, ***inertia matrices***, and ***angular momenta*** of the ***engines*** were also found in Unit 1.

$$ {}^R\underset{\sim}{\omega}_{E_i} = (\omega_1 + \omega_{E_i})\underset{\sim}{b}_1 + \omega_2\,\underset{\sim}{b}_2 + \omega_3\,\underset{\sim}{b}_3 \qquad (i=1,2) $$

$$ \left[I_{G_i}\right]_A = \begin{bmatrix} I_{x_b x_b}^{E} & 0 & 0 \\ 0 & I_{y_b y_b}^{E} & 0 \\ 0 & 0 & I_{z_b z_b}^{E} \end{bmatrix} \qquad (i=1,2) $$

$$ \underset{\sim}{H}_{G_i} = \left(I_{x_b x_b}^{E}(\omega_1 + \omega_{E_i})\right)\underset{\sim}{b}_1 + \left(I_{y_b y_b}^{E}\omega_2\right)\underset{\sim}{b}_2 + \left(I_{z_b z_b}^{E}\omega_3\right)\underset{\sim}{b}_3 \qquad (i=1,2) $$

Recall here that the inertias $I_{y_b y_b}^{E}$ and $I_{z_b z_b}^{E}$ are ***equal*** and they are ***constant*** relative to directions fixed in the ***airframe*** because of the assumed ***rotational symmetry*** of the engines about the $\underset{\sim}{b}_1$ direction.

Kinematics:

The velocities of the mass centers $G_i\ (i=1,2)$ and G_A, can all be written in terms of the velocity of G the mass center of the aircraft as follows.

$$ {}^R\underset{\sim}{v}_{G_i} = {}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_i/G} \qquad (i=1,2) $$

$$ {}^R\underset{\sim}{v}_{G_A} = {}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G} $$

Partial Angular Velocities and Partial Velocities:

Due to the ***convenient choice*** of ***generalized speeds***, the partial angular velocities and partial velocities take on particularly simple forms.

$$ \frac{\partial\,{}^R\underset{\sim}{\omega}_{E_i}}{\partial u_k} = \underset{\sim}{0} \qquad (k=1,2,3;\ i=1,2) $$

$$ \frac{\partial\,{}^R\underset{\sim}{\omega}_{E_i}}{\partial u_{k+3}} = \underset{\sim}{b}_k \qquad (k=1,2,3;\ i=1,2) $$

$$ \frac{\partial\,{}^R\underset{\sim}{\omega}_A}{\partial u_k} = \underset{\sim}{0} \qquad (k=1,2,3) $$

$$ \frac{\partial\,{}^R\underset{\sim}{\omega}_A}{\partial u_{k+3}} = \underset{\sim}{b}_k \qquad (k=1,2,3) $$

$$ \frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_k} = \underset{\sim}{b}_k \qquad (k=1,2,3) $$

$$ \frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_{k+3}} = \underset{\sim}{0} \qquad (k=1,2,3) $$

$$ \frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial u_k} = \frac{\partial}{\partial u_k}\left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_i/G}\right) = \frac{\partial\,{}^R\underset{\sim}{v}_G}{\partial u_k} = \underset{\sim}{b}_k \qquad (k=1,2,3;\ i=1,2) $$

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_{k+3}} = \frac{\partial}{\partial u_{k+3}}\left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_i/G}\right) = \frac{\partial\, {}^R\underset{\sim}{\omega}_A}{\partial u_{k+3}} \times \underset{\sim}{r}_{G_i/G} = \underset{\sim}{b}_k \times \underset{\sim}{r}_{G_i/G}} \quad (k=1,2,3;\, i=1,2)$$

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_A}}{\partial u_k} = \frac{\partial}{\partial u_k}\left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right) = \frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial u_k} = \underset{\sim}{b}_k} \quad (k=1,2,3)$$

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_A}}{\partial u_{k+3}} = \frac{\partial}{\partial u_{k+3}}\left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/G}\right) = \frac{\partial\, {}^R\underset{\sim}{\omega}_A}{\partial u_{k+3}} \times \underset{\sim}{r}_{G_A/G} = \underset{\sim}{b}_k \times \underset{\sim}{r}_{G_A/G}} \quad (k=1,2,3)$$

Generalized Forces:

The generalized forces associated with the six generalized speeds are

$$\boxed{F_{u_k} = \underset{\sim}{F}_{\text{ext}} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial u_k} + \underset{\sim}{T}^G_{\text{ext}} \cdot \underbrace{\frac{\partial\, {}^R\underset{\sim}{\omega}_A}{\partial u_k}}_{\text{zero}} = \left(\underset{\sim}{F}^A_{\text{ext}} + \sum_{i=1}^{2} \underset{\sim}{F}^{G_i}_{\text{ext}}\right) \cdot \underset{\sim}{b}_k} \quad (k=1,2,3)$$

$$\boxed{F_{u_{k+3}} = \underset{\sim}{F}_{\text{ext}} \cdot \underbrace{\frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial u_{k+3}}}_{\text{zero}} + \underset{\sim}{T}^G_{\text{ext}} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_A}{\partial u_{k+3}} = \left(\underset{\sim}{T}^A_{\text{ext}} + \sum_{i=1}^{2} \underset{\sim}{T}^{G_i}_{\text{ext}} + \left(\underset{\sim}{r}_{G_A/G} \times \underset{\sim}{F}^A_{\text{ext}}\right) + \sum_{i=1}^{2}\left(\underset{\sim}{r}_{G_i/G} \times \underset{\sim}{F}^{G_i}_{\text{ext}}\right)\right) \cdot \underset{\sim}{b}_k} \quad (k=1,2,3)$$

Kane's Equations:

For the airframe with two engines, Kane's equations can be written as

$$\boxed{\sum_{i=1}^{3}\left(m_i\, {}^R\underset{\sim}{a}_{G_i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right) + \sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i} \cdot {}^R\underset{\sim}{\alpha}_{B_i}\right) + \left({}^R\underset{\sim}{\omega}_{B_i} \times \underset{\sim}{H}_{G_i}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k} = F_{u_k}} \quad (k=1,\ldots,6)$$

Here, the number of bodies is $N_B = 3$, and G_3 represents G_A the mass center of the airframe. To find the equations of motion, consider first the sums on the left side of the equation.

Using the results for the partial velocities for the ***first three generalized speeds*** $\{u, v, w\}$, the sums on the left side of Kane's equations can be written as follows.

1. $$\sum_{i=1}^{3}\left(m_i\, {}^R\underset{\sim}{a}_{G_i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right) = \left(m_E\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_k}\right) + \left(m_E\, {}^R\underset{\sim}{a}_{G_2} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial u_k}\right) + \left(m_A\, {}^R\underset{\sim}{a}_{G_A} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_A}}{\partial u_k}\right)$$

$$= \left(m_E\, {}^R\underset{\sim}{a}_{G_1} \cdot \underset{\sim}{b}_k\right) + \left(m_E\, {}^R\underset{\sim}{a}_{G_2} \cdot \underset{\sim}{b}_k\right) + \left(m_A\, {}^R\underset{\sim}{a}_{G_A} \cdot \underset{\sim}{b}_k\right)$$

$$= \underbrace{\left(m_E\, {}^R\underset{\sim}{a}_{G_1} + m_E\, {}^R\underset{\sim}{a}_{G_2} + m_A\, {}^R\underset{\sim}{a}_{G_A}\right)}_{m_T\, {}^R\underset{\sim}{a}_G} \cdot \underset{\sim}{b}_k$$

$$\Rightarrow \boxed{\sum_{i=1}^{3}\left(m_i\, {}^R\underset{\sim}{a}_{G_i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right) = m_T\, {}^R\underset{\sim}{a}_G \cdot \underset{\sim}{b}_k} \quad (k=1,2,3)$$

2. $$\boxed{\sum_{i=1}^{3}\left[\left(\underset{\sim}{I}_{G_i}\cdot {}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times \underset{\sim}{H}_{G_i}\right)\right]\cdot\underbrace{\frac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}}_{\text{zero}}=0}\qquad (k=1,2,3)$$

Using the results for the partial velocities for the ***last three generalized speeds*** $\{\omega_1,\omega_2,\omega_3\}$, the sums on the left side of Kane's equations can be written as follows.

1. $$\sum_{i=1}^{3}\left(m_i\,{}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial u_{k+3}}\right)=\left(m_E\,{}^R\underset{\sim}{a}_{G_1}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_1}}{\partial u_{k+3}}\right)+\left(m_E\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_{k+3}}\right)+\left(m_A\,{}^R\underset{\sim}{a}_{G_A}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_A}}{\partial u_{k+3}}\right)$$

$$=\left(m_E\,{}^R\underset{\sim}{a}_{G_1}\cdot\left(\underset{\sim}{b}_k\times\underset{\sim}{r}_{G_1/G}\right)\right)+\left(m_E\,{}^R\underset{\sim}{a}_{G_2}\cdot\left(\underset{\sim}{b}_k\times\underset{\sim}{r}_{G_2/G}\right)\right)+\left(m_A\,{}^R\underset{\sim}{a}_{G_A}\cdot\left(\underset{\sim}{b}_k\times\underset{\sim}{r}_{G_A/G}\right)\right)$$

$$=\left(m_E\,\underset{\sim}{b}_k\cdot\left(\underset{\sim}{r}_{G_1/G}\times{}^R\underset{\sim}{a}_{G_1}\right)\right)+\left(m_E\,\underset{\sim}{b}_k\cdot\left(\underset{\sim}{r}_{G_2/G}\times{}^R\underset{\sim}{a}_{G_2}\right)\right)+\left(m_A\,\underset{\sim}{b}_k\cdot\left(\underset{\sim}{r}_{G_A/G}\times{}^R\underset{\sim}{a}_{G_A}\right)\right)$$

$$=\underset{\sim}{b}_k\cdot\left[m_E\left(\underset{\sim}{r}_{G_1/G}\times{}^R\underset{\sim}{a}_{G_1}\right)+m_E\left(\underset{\sim}{r}_{G_2/G}\times{}^R\underset{\sim}{a}_{G_2}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\times{}^R\underset{\sim}{a}_{G_A}\right)\right]$$

Here, the accelerations of the individual mass centers can be written in terms of the acceleration of G the mass center of the aircraft. Consider the first term in the square brackets.

$$m_E\left(\underset{\sim}{r}_{G_1/G}\times{}^R\underset{\sim}{a}_{G_1}\right)=m_E\,\underset{\sim}{r}_{G_1/G}\times\left[{}^R\underset{\sim}{a}_G+\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_1/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)\right]$$

$$=\left(\underset{\sim}{r}_{G_1/G}\times m_E\,{}^R\underset{\sim}{a}_G\right)+\underset{\sim}{r}_{G_1/G}\times m_E\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_1/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)\right]$$

Similar results are true for the second and third terms in the square brackets. Summing those three terms gives

$$m_E\left(\underset{\sim}{r}_{G_1/G}\times{}^R\underset{\sim}{a}_{G_1}\right)+m_E\left(\underset{\sim}{r}_{G_2/G}\times{}^R\underset{\sim}{a}_{G_2}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\times{}^R\underset{\sim}{a}_{G_A}\right)$$

$$=\left(\underset{\sim}{r}_{G_1/G}\times m_E\,{}^R\underset{\sim}{a}_G\right)+m_E\,\underset{\sim}{r}_{G_1/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_1/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)\right]$$

$$+\left(\underset{\sim}{r}_{G_2/G}\times m_E\,{}^R\underset{\sim}{a}_G\right)+m_E\,\underset{\sim}{r}_{G_2/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_2/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)\right]$$

$$+\left(\underset{\sim}{r}_{G_A/G}\times m_A\,{}^R\underset{\sim}{a}_G\right)+m_A\,\underset{\sim}{r}_{G_A/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_A/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\right]$$

$$=\underbrace{\left(m_E\,\underset{\sim}{r}_{G_1/G}+m_E\,\underset{\sim}{r}_{G_2/G}+m_A\,\underset{\sim}{r}_{G_A/G}\right)}_{\text{zero}}\times{}^R\underset{\sim}{a}_G+m_E\,\underset{\sim}{r}_{G_1/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_1/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)\right]$$

$$+m_E\,\underset{\sim}{r}_{G_2/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_2/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)\right]$$

$$+m_A\,\underset{\sim}{r}_{G_A/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_A/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\right]$$

Substituting these results into the above equation gives

$$\boxed{\begin{aligned}\sum_{i=1}^{3}\left(m_i\,{}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial u_{k+3}}\right) &= \underset{\sim}{b}_k\cdot\left\{m_E\,\underset{\sim}{r}_{G_1/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_1/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)\right]\right\}\\ &+\underset{\sim}{b}_k\cdot\left\{m_E\,\underset{\sim}{r}_{G_2/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_2/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)\right]\right\}\\ &+\underset{\sim}{b}_k\cdot\left\{m_A\,\underset{\sim}{r}_{G_A/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_A/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\right]\right\}\end{aligned}}\qquad (k=1,2,3)$$

2. $$\boxed{\sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{B_i}}{\partial u_{k+3}}=\sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\underset{\sim}{b}_k}\qquad (k=1,2,3)$$

Combining the above results, the left side of Kane's equations for the ***first three generalized speeds*** $\{u,v,w\}$ are

$$\boxed{\left(m_T\,{}^R\underset{\sim}{a}_G\right).\,\underset{\sim}{b}_k=\left(\underset{\sim}{F}^A_{\text{ext}}+\sum_{i=1}^{2}\underset{\sim}{F}^{G_i}_{\text{ext}}\right)\cdot\underset{\sim}{b}_k}\qquad (k=1,2,3)$$

where

$${}^R\underset{\sim}{a}_G=\frac{{}^Rd}{dt}\left({}^R\underset{\sim}{v}_G\right)=\frac{{}^Ad}{dt}\left({}^R\underset{\sim}{v}_G\right)+{}^R\underset{\sim}{\omega}_A\times{}^R\underset{\sim}{v}_G=\left(\dot{u}\,\underset{\sim}{b}_1+\dot{v}\,\underset{\sim}{b}_2+\dot{w}\,\underset{\sim}{b}_3\right)+\begin{vmatrix}\underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3\\ \omega_1 & \omega_2 & \omega_3\\ u & v & w\end{vmatrix}$$

$$\Rightarrow\boxed{{}^R\underset{\sim}{a}_G=\left(\dot{u}+\omega_2 w-\omega_3 v\right)\underset{\sim}{b}_1+\left(\dot{v}+\omega_3 u-\omega_1 w\right)\underset{\sim}{b}_2+\left(\dot{w}+\omega_1 v-\omega_2 u\right)\underset{\sim}{b}_3}$$

These equations are obviously just a statement of ***Newton's second law*** for the aircraft written along the ***body-fixed directions***.

The left side of Kane's equations for the ***last three generalized speeds*** $\{\omega_1,\omega_2,\omega_3\}$ can be written as follows. In the simplification process, advantage is taken of the vector identity $\boxed{\underset{\sim}{a}\times\left(\underset{\sim}{b}\times\underset{\sim}{c}\right)=\left(\underset{\sim}{a}\cdot\underset{\sim}{c}\right)\underset{\sim}{b}-\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)\underset{\sim}{c}}$

$$\begin{aligned}&\sum_{i=1}^{3}\left(m_i\,{}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right)+\sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}\\ &=\left[\left(\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\alpha}_{E_1}\right)+\left({}^R\underset{\sim}{\omega}_{E_1}\times\underset{\sim}{H}_{G_1}\right)+m_E\,\underset{\sim}{r}_{G_1/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_1/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_1/G}\right)\right]\right]\cdot\underset{\sim}{b}_k\\ &\quad+\left[\left(\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\alpha}_{E_2}\right)+\left({}^R\underset{\sim}{\omega}_{E_2}\times\underset{\sim}{H}_{G_2}\right)+m_E\,\underset{\sim}{r}_{G_2/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_2/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_2/G}\right)\right]\right]\cdot\underset{\sim}{b}_k\\ &\quad+\left[\left(\underset{\approx}{I}_{G_A}\cdot{}^R\underset{\sim}{\alpha}_A\right)+\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{H}_{G_A}\right)+m_A\,\underset{\sim}{r}_{G_A/G}\times\left[\left({}^R\underset{\sim}{\alpha}_A\times\underset{\sim}{r}_{G_A/G}\right)+{}^R\underset{\sim}{\omega}_A\times\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{r}_{G_A/G}\right)\right]\right]\cdot\underset{\sim}{b}_k\end{aligned}$$

$$
=\left[\left(\underset{\approx}{I}_{G_1}\cdot{}^{R}\underset{\sim}{\alpha}_{E_1}\right)+\left(m_E\,\underset{\sim}{r}_{G_1/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_1/G}\right)\right)\right]\cdot\underset{\sim}{b}_k+\left[\left(\underset{\approx}{I}_{G_2}\cdot{}^{R}\underset{\sim}{\alpha}_{E_2}\right)+\left(m_E\,\underset{\sim}{r}_{G_2/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_2/G}\right)\right)\right]\cdot\underset{\sim}{b}_k
$$

$$
+\left[\left(\underset{\approx}{I}_{G_A}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right)+\left(m_A\,\underset{\sim}{r}_{G_A/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_A/G}\right)\right)\right]\cdot\underset{\sim}{b}_k+\left[\left({}^{R}\underset{\sim}{\omega}_{E_1}\times\underset{\sim}{H}_{G_1}\right)+\left({}^{R}\underset{\sim}{\omega}_{E_2}\times\underset{\sim}{H}_{G_2}\right)+\left({}^{R}\underset{\sim}{\omega}_{A}\times\underset{\sim}{H}_{G_A}\right)\right]\cdot\underset{\sim}{b}_k
$$

$$
+\sum_{i=1}^{2}\left[m_E\,\underset{\sim}{r}_{G_i/G}\times\left[\left(\underset{\sim}{r}_{G_i/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right){}^{R}\underset{\sim}{\omega}_{A}-{}^{R}\omega_A^2\,\underset{\sim}{r}_{G_i/G}\right]\right]\cdot\underset{\sim}{b}_k
$$

$$
+\left[m_A\,\underset{\sim}{r}_{G_A/G}\times\left[\left(\underset{\sim}{r}_{G_A/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right){}^{R}\underset{\sim}{\omega}_{A}-{}^{R}\omega_A^2\,\underset{\sim}{r}_{G_A/G}\right]\right]\cdot\underset{\sim}{b}_k
$$

$$
\boxed{\begin{aligned}
&\sum_{i=1}^{3}\left(m_i\,{}^{R}\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^{R}\underset{\sim}{v}_{G_i}}{\partial u_k}\right)+\sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^{R}\underset{\sim}{\alpha}_{B_i}\right)+\left({}^{R}\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_{B_i}}{\partial u_k}\\
&\quad=\left[\sum_{i=1}^{2}\left(\underset{\approx}{I}_{G_i}\cdot{}^{R}\underset{\sim}{\alpha}_{E_i}\right)+\left(m_E\,\underset{\sim}{r}_{G_i/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_i/G}\right)\right)\right]\cdot\underset{\sim}{b}_k+\left[\left(\underset{\approx}{I}_{G_A}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right)+\left(m_A\,\underset{\sim}{r}_{G_A/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_A/G}\right)\right)\right]\cdot\underset{\sim}{b}_k\\
&\qquad+\sum_{i=1}^{2}\left[\left({}^{R}\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_i/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\right]\cdot\underset{\sim}{b}_k\\
&\qquad+\left[\left({}^{R}\underset{\sim}{\omega}_{A}\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_A/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\right]\cdot\underset{\sim}{b}_k
\end{aligned}}
$$

This result can be simplified by making use of results presented in Unit 2 of this volume. Specifically,

$$
\boxed{\left(\underset{\approx}{I}_{G_A}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right)+m_A\,\underset{\sim}{r}_{G_A/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_A/G}\right)=\left(\underset{\approx}{I}_{G}\right)_A\cdot{}^{R}\underset{\sim}{\alpha}_{A}}
$$

$$
\boxed{\underset{\approx}{I}_{G_i}\cdot{}^{R}\underset{\sim}{\alpha}_{E_i}+m_E\,\underset{\sim}{r}_{G_i/G}\times\left({}^{R}\underset{\sim}{\alpha}_{A}\times\underset{\sim}{r}_{G_i/G}\right)=\left[\left(\underset{\approx}{I}_{G}\right)_{E_i}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right]+\left[\underset{\approx}{I}_{G_i}\cdot{}^{R}\underset{\sim}{\hat{\alpha}}_{E_i}\right]}\qquad(i=1,2)
$$

$$
\boxed{{}^{R}\underset{\sim}{\alpha}_{E_i}={}^{R}\underset{\sim}{\alpha}_{A}+{}^{R}\underset{\sim}{\hat{\alpha}}_{E_i}}\quad\text{with}\quad\boxed{{}^{R}\underset{\sim}{\hat{\alpha}}_{E_i}=\dot{\omega}_{E_i}\,\underset{\sim}{b}_1+\omega_3\omega_{E_i}\,\underset{\sim}{b}_2-\omega_2\omega_{E_i}\,\underset{\sim}{b}_3}\qquad(i=1,2)
$$

Substituting these results into the above equation and simplifying gives

$$
\sum_{i=1}^{3}\left(m_i\,{}^{R}\underset{\sim}{a}_{G_i}\cdot\frac{\partial\,{}^{R}\underset{\sim}{v}_{G_i}}{\partial u_k}\right)+\sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i}\cdot{}^{R}\underset{\sim}{\alpha}_{B_i}\right)+\left({}^{R}\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_{B_i}}{\partial u_k}
$$

$$
=\left[\left[\left(\underset{\approx}{I}_{G}\right)_{E_1}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right]+\left[\underset{\approx}{I}_{G_1}\cdot{}^{R}\underset{\sim}{\hat{\alpha}}_{E_1}\right]\right]\cdot\underset{\sim}{b}_k+\left[\left[\left(\underset{\approx}{I}_{G}\right)_{E_2}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right]+\left[\underset{\approx}{I}_{G_2}\cdot{}^{R}\underset{\sim}{\hat{\alpha}}_{E_2}\right]\right]\cdot\underset{\sim}{b}_k
$$

$$
+\left[\left(\underset{\approx}{I}_{G}\right)_{A}\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right]\cdot\underset{\sim}{b}_k+\sum_{i=1}^{2}\left[\left({}^{R}\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_i/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\right]\cdot\underset{\sim}{b}_k
$$

$$
+\left[\left({}^{R}\underset{\sim}{\omega}_{A}\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_A/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\right]\cdot\underset{\sim}{b}_k
$$

$$
=\left[\left[\left(\underset{\approx}{I}_{G}\right)_{E_1}+\left(\underset{\approx}{I}_{G}\right)_{E_2}+\left(\underset{\approx}{I}_{G}\right)_{A}\right]\cdot{}^{R}\underset{\sim}{\alpha}_{A}\right]\cdot\underset{\sim}{b}_k+\left[\left[\underset{\approx}{I}_{G_1}\cdot{}^{R}\underset{\sim}{\hat{\alpha}}_{E_1}\right]+\left[\underset{\approx}{I}_{G_2}\cdot{}^{R}\underset{\sim}{\hat{\alpha}}_{E_2}\right]\right]\cdot\underset{\sim}{b}_k
$$

$$
+\sum_{i=1}^{2}\left[\left({}^{R}\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_i/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\right]\cdot\underset{\sim}{b}_k
$$

$$
+\left[\left({}^{R}\underset{\sim}{\omega}_{A}\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_A/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\right]\cdot\underset{\sim}{b}_k
$$

$$\Rightarrow \boxed{\begin{aligned}&\sum_{i=1}^{3}\left(m_i\, {}^R\underset{\sim}{a}_{G_i}\cdot\frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right)+\sum_{i=1}^{3}\left[\left(\underset{\approx}{I}_{G_i}\cdot {}^R\underset{\sim}{\alpha}_{B_i}\right)+\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}\\ &\quad=\left(\left[\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot {}^R\underset{\sim}{\alpha}_A\right]+\left[\underset{\approx}{I}_{G_1}\cdot {}^R\hat{\underset{\sim}{\alpha}}_{E_1}\right]+\left[\underset{\approx}{I}_{G_2}\cdot {}^R\hat{\underset{\sim}{\alpha}}_{E_2}\right]\right)\cdot\underset{\sim}{b}_k\\ &\qquad+\sum_{i=1}^{2}\left[\left({}^R\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot {}^R\underset{\sim}{\omega}_A\right)\left(\underset{\sim}{r}_{G_i/G}\times {}^R\underset{\sim}{\omega}_A\right)\right]\cdot\underset{\sim}{b}_k\\ &\qquad+\left[\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot {}^R\underset{\sim}{\omega}_A\right)\left(\underset{\sim}{r}_{G_A/G}\times {}^R\underset{\sim}{\omega}_A\right)\right]\cdot\underset{\sim}{b}_k\end{aligned}}$$

Substituting into Kane's equations for the ***last three generalized speeds*** gives

$$\boxed{\begin{aligned}&\left(\left[\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot {}^R\underset{\sim}{\alpha}_A\right]+\left[\underset{\approx}{I}_{G_1}\cdot {}^R\hat{\underset{\sim}{\alpha}}_{E_1}\right]+\left[\underset{\approx}{I}_{G_2}\cdot {}^R\hat{\underset{\sim}{\alpha}}_{E_2}\right]\right)\cdot\underset{\sim}{b}_k\\ &\quad+\sum_{i=1}^{2}\left[\left({}^R\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot {}^R\underset{\sim}{\omega}_A\right)\left(\underset{\sim}{r}_{G_i/G}\times {}^R\underset{\sim}{\omega}_A\right)\right]\cdot\underset{\sim}{b}_k\\ &\qquad+\left[\left({}^R\underset{\sim}{\omega}_A\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot {}^R\underset{\sim}{\omega}_A\right)\left(\underset{\sim}{r}_{G_A/G}\times {}^R\underset{\sim}{\omega}_A\right)\right]\cdot\underset{\sim}{b}_k=F_{u_{k+3}}\end{aligned}}\qquad (k=1,2,3)$$

Consider now the ***expansion*** of each of the terms in these equations into ***component form***.

1. $\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot {}^R\underset{\sim}{\alpha}_A \Rightarrow \begin{bmatrix} I^G_{x_bx_b} & 0 & -I^G_{x_bz_b}\\ 0 & I^G_{y_by_b} & 0\\ -I^G_{x_bz_b} & 0 & I^G_{z_bz_b}\end{bmatrix}\begin{Bmatrix}\dot\omega_1\\ \dot\omega_2\\ \dot\omega_3\end{Bmatrix}$

$$\Rightarrow \boxed{\left(\underset{\approx}{I}_G\right)_{\text{aircraft}}\cdot {}^R\underset{\sim}{\alpha}_A=\left(I^G_{x_bx_b}\dot\omega_1-I^G_{x_bz_b}\dot\omega_3\right)\underset{\sim}{b}_1+\left(I^G_{y_by_b}\dot\omega_2\right)\underset{\sim}{b}_2+\left(I^G_{z_bz_b}\dot\omega_3-I^G_{x_bz_b}\dot\omega_1\right)\underset{\sim}{b}_3}$$

2. $\underset{\approx}{I}_{G_i}\cdot {}^R\hat{\underset{\sim}{\alpha}}_{E_i} \Rightarrow \begin{bmatrix} I^E_{x_bx_b} & 0 & 0\\ 0 & I^E_{y_by_b} & 0\\ 0 & 0 & I^E_{z_bz_b}\end{bmatrix}\begin{Bmatrix}\dot\omega_{E_i}\\ \omega_3\omega_{E_i}\\ -\omega_2\omega_{E_i}\end{Bmatrix}$

$$\Rightarrow \boxed{\underset{\approx}{I}_{G_i}\cdot {}^R\hat{\underset{\sim}{\alpha}}_{E_i}=\left(I^E_{x_bx_b}\dot\omega_{E_i}\right)\underset{\sim}{b}_1+\left(I^E_{y_by_b}\omega_3\omega_{E_i}\right)\underset{\sim}{b}_2-\left(I^E_{z_bz_b}\omega_2\omega_{E_i}\right)\underset{\sim}{b}_3}\qquad (i=1,2)$$

3. ${}^R\underset{\sim}{\omega}_A\times\underset{\sim}{H}_{G_A}=\begin{vmatrix}\underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3\\ \omega_1 & \omega_2 & \omega_3\\ I^{G_A}_{x_bx_b}\omega_1-I^{G_A}_{x_bz_b}\omega_3 & I^{G_A}_{y_by_b}\omega_2 & I^{G_A}_{z_bz_b}\omega_3-I^{G_A}_{x_bz_b}\omega_1\end{vmatrix}$

$$=\left[\left(I^{G_A}_{z_bz_b}\omega_3-I^{G_A}_{x_bz_b}\omega_1\right)\omega_2-I^{G_A}_{y_by_b}\omega_2\omega_3\right]\underset{\sim}{b}_1+\left[\left(I^{G_A}_{x_bx_b}\omega_1-I^{G_A}_{x_bz_b}\omega_3\right)\omega_3-\left(I^{G_A}_{z_bz_b}\omega_3-I^{G_A}_{x_bz_b}\omega_1\right)\omega_1\right]\underset{\sim}{b}_2$$
$$+\left[I^{G_A}_{y_by_b}\omega_1\omega_2-\left(I^{G_A}_{x_bx_b}\omega_1-I^{G_A}_{x_bz_b}\omega_3\right)\omega_2\right]\underset{\sim}{b}_3$$

$$\Rightarrow \boxed{\begin{aligned}{}^R\underset{\sim}{\omega}_A\times\underset{\sim}{H}_{G_A}&=\left[\left(I^{G_A}_{z_bz_b}-I^{G_A}_{y_by_b}\right)\omega_2\omega_3-I^{G_A}_{x_bz_b}\omega_1\omega_2\right]\underset{\sim}{b}_1+\left[I^{G_A}_{x_bz_b}\left(\omega_1^2-\omega_3^2\right)+\left(I^{G_A}_{x_bx_b}-I^{G_A}_{z_bz_b}\right)\omega_1\omega_3\right]\underset{\sim}{b}_2\\ &\quad+\left[\left(I^{G_A}_{y_by_b}-I^{G_A}_{x_bx_b}\right)\omega_1\omega_2+I^{G_A}_{x_bz_b}\omega_2\omega_3\right]\underset{\sim}{b}_3\end{aligned}}$$

4. $${}^{R}\underset{\sim}{\omega}_{E_i} \times \underset{\sim}{H}_{G_i} = \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_1 + \omega_{E_i} & \omega_2 & \omega_3 \\ I^E_{x_b x_b}\left(\omega_1 + \omega_{E_i}\right) & I^E_{y_b y_b}\omega_2 & I^E_{z_b z_b}\omega_3 \end{vmatrix}$$

$$= \left[\left(I^E_{z_b z_b} - I^E_{y_b y_b}\right)\omega_2\omega_3\right]\underset{\sim}{b}_1 + \left[I^E_{x_b x_b}\left(\omega_1 + \omega_{E_i}\right)\omega_3 - I^E_{z_b z_b}\omega_3\left(\omega_1 + \omega_{E_i}\right)\right]\underset{\sim}{b}_2$$
$$+ \left[I^E_{y_b y_b}\omega_2\left(\omega_1 + \omega_{E_i}\right) - I^E_{x_b x_b}\left(\omega_1 + \omega_{E_i}\right)\omega_2\right]\underset{\sim}{b}_3$$

$$\Rightarrow \boxed{\begin{aligned} {}^{R}\underset{\sim}{\omega}_{E_i} \times \underset{\sim}{H}_{G_i} &= \left[\left(I^E_{z_b z_b} - I^E_{y_b y_b}\right)\omega_2\omega_3\right]\underset{\sim}{b}_1 + \left[\left(I^E_{x_b x_b} - I^E_{z_b z_b}\right)\left(\omega_1 + \omega_{E_i}\right)\omega_3\right]\underset{\sim}{b}_2 \\ &\quad + \left[\left(I^E_{y_b y_b} - I^E_{x_b x_b}\right)\left(\omega_1 + \omega_{E_i}\right)\omega_2\right]\underset{\sim}{b}_3 \end{aligned}}$$

5. $$m_A\left({}^{R}\underset{\sim}{\omega}_A \cdot \underset{\sim}{r}_{G_A/G}\right)\left(\underset{\sim}{r}_{G_A/G} \times {}^{R}\underset{\sim}{\omega}_A\right) = m_A\left(x_A\omega_1 + z_A\omega_3\right)\begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ x_A & 0 & z_A \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$\Rightarrow \boxed{m_A\left({}^{R}\underset{\sim}{\omega}_A \cdot \underset{\sim}{r}_{G_A/G}\right)\left(\underset{\sim}{r}_{G_A/G} \times {}^{R}\underset{\sim}{\omega}_A\right) = m_A\left(x_A\omega_1 + z_A\omega_3\right)\left[\left(-z_A\omega_2\right)\underset{\sim}{b}_1 + \left(z_A\omega_1 - x_A\omega_3\right)\underset{\sim}{b}_2 + \left(x_A\omega_2\right)\underset{\sim}{b}_3\right]}$$

6. $$m_E\left({}^{R}\underset{\sim}{\omega}_A \cdot \underset{\sim}{r}_{G_1/G}\right)\left(\underset{\sim}{r}_{G_1/G} \times {}^{R}\underset{\sim}{\omega}_A\right) = m_E\left(x_E\omega_1 + y_E\omega_2 + z_E\omega_3\right)\begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ x_E & y_E & z_E \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} &m_E\left({}^{R}\underset{\sim}{\omega}_A \cdot \underset{\sim}{r}_{G_1/G}\right)\left(\underset{\sim}{r}_{G_1/G} \times {}^{R}\underset{\sim}{\omega}_A\right) \\ &\quad = m_E\left(x_E\omega_1 + y_E\omega_2 + z_E\omega_3\right)\left[\left(y_E\omega_3 - z_E\omega_2\right)\underset{\sim}{b}_1 + \left(z_E\omega_1 - x_E\omega_3\right)\underset{\sim}{b}_2 + \left(x_E\omega_2 - y_E\omega_1\right)\underset{\sim}{b}_3\right] \end{aligned}}$$

7. $$m_E\left({}^{R}\underset{\sim}{\omega}_A \cdot \underset{\sim}{r}_{G_2/G}\right)\left(\underset{\sim}{r}_{G_2/G} \times {}^{R}\underset{\sim}{\omega}_A\right) = m_E\left(x_E\omega_1 - y_E\omega_2 + z_E\omega_3\right)\begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ x_E & -y_E & z_E \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} &m_E\left({}^{R}\underset{\sim}{\omega}_A \cdot \underset{\sim}{r}_{G_2/G}\right)\left(\underset{\sim}{r}_{G_2/G} \times {}^{R}\underset{\sim}{\omega}_A\right) \\ &\quad = m_E\left(x_E\omega_1 - y_E\omega_2 + z_E\omega_3\right)\left[\left(-y_E\omega_3 - z_E\omega_2\right)\underset{\sim}{b}_1 + \left(z_E\omega_1 - x_E\omega_3\right)\underset{\sim}{b}_2 + \left(x_E\omega_2 + y_E\omega_1\right)\underset{\sim}{b}_3\right] \end{aligned}}$$

Summing the last two terms gives the following.

$$8.\quad m_E\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{r}_{G_1/G}\right)\left(\underset{\sim}{r}_{G_1/G}\times{}^R\underset{\sim}{\omega}_A\right)+m_E\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{r}_{G_2/G}\right)\left(\underset{\sim}{r}_{G_2/G}\times{}^R\underset{\sim}{\omega}_A\right)$$
$$=\left[m_E\left(x_E\omega_1+y_E\omega_2+z_E\omega_3\right)\left(y_E\omega_3-z_E\omega_2\right)+m_E\left(x_E\omega_1-y_E\omega_2+z_E\omega_3\right)\left(-y_E\omega_3-z_E\omega_2\right)\right]\underset{\sim}{b}_1$$
$$+\left[m_E\left(x_E\omega_1+y_E\omega_2+z_E\omega_3\right)\left(z_E\omega_1-x_E\omega_3\right)+m_E\left(x_E\omega_1-y_E\omega_2+z_E\omega_3\right)\left(z_E\omega_1-x_E\omega_3\right)\right]\underset{\sim}{b}_2$$
$$+\left[m_E\left(x_E\omega_1+y_E\omega_2+z_E\omega_3\right)\left(x_E\omega_2-y_E\omega_1\right)+m_E\left(x_E\omega_1-y_E\omega_2+z_E\omega_3\right)\left(x_E\omega_2+y_E\omega_1\right)\right]\underset{\sim}{b}_3$$

$$\boxed{\begin{aligned}&m_E\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{r}_{G_1/G}\right)\left(\underset{\sim}{r}_{G_1/G}\times{}^R\underset{\sim}{\omega}_A\right)+m_E\left({}^R\underset{\sim}{\omega}_A\cdot\underset{\sim}{r}_{G_2/G}\right)\left(\underset{\sim}{r}_{G_2/G}\times{}^R\underset{\sim}{\omega}_A\right)\\&\quad=m_E\left[-2x_Ez_E\omega_1\omega_2+2\left(y_E^2-z_E^2\right)\omega_2\omega_3\right]\underset{\sim}{b}_1+m_E\left[2x_Ez_E\left(\omega_1^2-\omega_3^2\right)+2\left(z_E^2-x_E^2\right)\omega_1\omega_3\right]\underset{\sim}{b}_2\\&\quad\quad+m_E\left[2\left(x_E^2-y_E^2\right)\omega_1\omega_2+2x_Ez_E\omega_2\omega_3\right]\underset{\sim}{b}_3\end{aligned}}$$

Substituting these results into the ***engine terms*** on the left side of the moment equation gives

$$\sum_{i=1}^{2}\left[\left({}^R\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot{}^R\underset{\sim}{\omega}_A\right)\left(\underset{\sim}{r}_{G_i/G}\times{}^R\underset{\sim}{\omega}_A\right)\right]$$
$$=\left[\left(I^E_{z_bz_b}-I^E_{y_by_b}\right)\omega_2\omega_3\right]\underset{\sim}{b}_1+\left[\left(I^E_{x_bx_b}-I^E_{z_bz_b}\right)\left(\omega_1+\omega_{E_1}\right)\omega_3\right]\underset{\sim}{b}_2+\left[\left(I^E_{y_by_b}-I^E_{x_bx_b}\right)\left(\omega_1+\omega_{E_1}\right)\omega_2\right]\underset{\sim}{b}_3$$
$$+\left[\left(I^E_{z_bz_b}-I^E_{y_by_b}\right)\omega_2\omega_3\right]\underset{\sim}{b}_1+\left[\left(I^E_{x_bx_b}-I^E_{z_bz_b}\right)\left(\omega_1+\omega_{E_2}\right)\omega_3\right]\underset{\sim}{b}_2+\left[\left(I^E_{y_by_b}-I^E_{x_bx_b}\right)\left(\omega_1+\omega_{E_2}\right)\omega_2\right]\underset{\sim}{b}_3$$
$$+m_E\left[-2x_Ez_E\omega_1\omega_2+2\left(y_E^2-z_E^2\right)\omega_2\omega_3\right]\underset{\sim}{b}_1+m_E\left[2x_Ez_E\left(\omega_1^2-\omega_3^2\right)+2\left(z_E^2-x_E^2\right)\omega_1\omega_3\right]\underset{\sim}{b}_2$$
$$+m_E\left[2\left(x_E^2-y_E^2\right)\omega_1\omega_2+2x_Ez_E\omega_2\omega_3\right]\underset{\sim}{b}_3$$
$$=\left[2\left(\left[I^E_{z_bz_b}+m_Ex_E^2+m_Ey_E^2\right]-\left[I^E_{y_by_b}+m_Ex_E^2+m_Ez_E^2\right]\right)\omega_2\omega_3-2m_Ex_Ez_E\omega_1\omega_2\right]\underset{\sim}{b}_1$$
$$+\left[2\left(\left[I^E_{x_bx_b}+m_Ey_E^2+m_Ez_E^2\right]-\left[I^E_{z_bz_b}+m_Ex_E^2+m_Ey_E^2\right]\right)\omega_1\omega_3+2m_Ex_Ez_E\left(\omega_1^2-\omega_3^2\right)\right]\underset{\sim}{b}_2$$
$$+\left[\left(I^E_{x_bx_b}-I^E_{z_bz_b}\right)\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3\right]\underset{\sim}{b}_2$$
$$+\left[2\left(\left[I^E_{y_by_b}+m_Ex_E^2+m_Ez_E^2\right]-\left[I^E_{x_bx_b}+m_Ey_E^2+m_Ez_E^2\right]\right)\omega_1\omega_2+2m_Ex_Ez_E\omega_2\omega_3\right]\underset{\sim}{b}_3$$
$$+\left[\left(I^E_{y_by_b}-I^E_{x_bx_b}\right)\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2\right]\underset{\sim}{b}_3$$

$$\Rightarrow\boxed{\begin{aligned}&\sum_{i=1}^{2}\left[\left({}^R\underset{\sim}{\omega}_{E_i}\times\underset{\sim}{H}_{G_i}\right)+m_E\left(\underset{\sim}{r}_{G_i/G}\cdot{}^R\underset{\sim}{\omega}_A\right)\left(\underset{\sim}{r}_{G_i/G}\times{}^R\underset{\sim}{\omega}_A\right)\right]\\&\quad=\left[2\left(\left(I^G_{z_bz_b}\right)_E-\left(I^G_{y_by_b}\right)_E\right)\omega_2\omega_3-2\left(I^G_{x_bz_b}\right)_E\omega_1\omega_2\right]\underset{\sim}{b}_1\\&\quad\quad+\left[2\left(\left(I^G_{x_bx_b}\right)_E-\left(I^G_{z_bz_b}\right)_E\right)\omega_1\omega_3+2\left(I^G_{x_bz_b}\right)_E\left(\omega_1^2-\omega_3^2\right)+\left(I^E_{x_bx_b}-I^E_{z_bz_b}\right)\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3\right]\underset{\sim}{b}_2\\&\quad\quad+\left[2\left(\left(I^G_{y_by_b}\right)_E-\left(I^G_{x_bx_b}\right)_E\right)\omega_1\omega_2+2\left(I^G_{x_bz_b}\right)_E\omega_2\omega_3+\left(I^E_{y_by_b}-I^E_{x_bx_b}\right)\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2\right]\underset{\sim}{b}_3\end{aligned}}$$

And substituting into the ***airframe terms*** gives

$$\left({}^{R}\underset{\sim}{\omega}_{A}\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_A/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)$$
$$=\left[\left(I_{z_bz_b}^{G_A}-I_{y_by_b}^{G_A}\right)\omega_2\omega_3-I_{x_bz_b}^{G_A}\omega_1\omega_2\right]\underset{\sim}{b}_1+\left[I_{x_bz_b}^{G_A}\left(\omega_1^2-\omega_3^2\right)+\left(I_{x_bx_b}^{G_A}-I_{z_bz_b}^{G_A}\right)\omega_1\omega_3\right]\underset{\sim}{b}_2$$
$$+\left[\left(I_{y_by_b}^{G_A}-I_{x_bx_b}^{G_A}\right)\omega_1\omega_2+I_{x_bz_b}^{G_A}\omega_2\omega_3\right]\underset{\sim}{b}_3$$
$$+m_A\left(x_A\omega_1+z_A\omega_3\right)\left[\left(-z_A\omega_2\right)\underset{\sim}{b}_1+\left(z_A\omega_1-x_A\omega_3\right)\underset{\sim}{b}_2+\left(x_A\omega_2\right)\underset{\sim}{b}_3\right]$$

$$=\left[\left(\left[I_{z_bz_b}^{G_A}+m_Ax_A^2\right]-\left[I_{y_by_b}^{G_A}+m_Ax_A^2+m_Az_A^2\right]\right)\omega_2\omega_3-\left(I_{x_bz_b}^{G_A}+m_Ax_Az_A\right)\omega_1\omega_2\right]\underset{\sim}{b}_1$$
$$+\left[\left(\left[I_{x_bx_b}^{G_A}+m_Az_A^2\right]-\left[I_{z_bz_b}^{G_A}+m_Ax_A^2\right]\right)\omega_1\omega_3+\left(I_{x_bz_b}^{G_A}+m_Ax_Az_A\right)\left(\omega_1^2-\omega_3^2\right)\right]\underset{\sim}{b}_2$$
$$+\left[\left(\left[I_{y_by_b}^{G_A}+m_Ax_A^2+m_Az_A^2\right]-\left[I_{x_bx_b}^{G_A}+m_Az_A^2\right]\right)\omega_1\omega_2+\left(I_{x_bz_b}^{G_A}+m_Ax_Az_A\right)\omega_2\omega_3\right]\underset{\sim}{b}_3$$

$$\boxed{\begin{aligned}&\left({}^{R}\underset{\sim}{\omega}_{A}\times\underset{\sim}{H}_{G_A}\right)+m_A\left(\underset{\sim}{r}_{G_A/G}\cdot{}^{R}\underset{\sim}{\omega}_{A}\right)\left(\underset{\sim}{r}_{G_A/G}\times{}^{R}\underset{\sim}{\omega}_{A}\right)\\&\quad=\left[\left(\left(I_{z_bz_b}^{G}\right)_A-\left(I_{y_by_b}^{G}\right)_A\right)\omega_2\omega_3-\left(I_{x_bz_b}^{G}\right)_A\omega_1\omega_2\right]\underset{\sim}{b}_1+\left[\left(\left(I_{x_bx_b}^{G}\right)_A-\left(I_{z_bz_b}^{G}\right)_A\right)\omega_1\omega_3+\left(I_{x_bz_b}^{G}\right)_A\left(\omega_1^2-\omega_3^2\right)\right]\underset{\sim}{b}_2\\&\qquad+\left[\left(\left(I_{y_by_b}^{G}\right)_A-\left(I_{x_bx_b}^{G}\right)_A\right)\omega_1\omega_2+\left(I_{x_bz_b}^{G}\right)_A\omega_2\omega_3\right]\underset{\sim}{b}_3\end{aligned}}$$

Finally, substituting again into Kane's equations for the last three generalized speeds and separating into individual equations gives

$k=1$:

$$\boxed{I_{x_bx_b}^{G}\dot{\omega}_1-I_{x_bz_b}^{G}\dot{\omega}_3+I_{x_bx_b}^{E}\dot{\omega}_{E_1}+I_{x_bx_b}^{E}\dot{\omega}_{E_2}+\left(I_{z_bz_b}^{G}-I_{y_by_b}^{G}\right)\omega_2\omega_3-I_{x_bz_b}^{G}\omega_1\omega_2=\underset{\sim}{T}_{\text{ext}}^{G}\cdot\underset{\sim}{b}_1}$$

$k=2$: Advantage is taken of the assumption that $I_{y_by_b}^{E}=I_{z_bz_b}^{E}$.

$$I_{y_by_b}^{G}\dot{\omega}_2+I_{y_by_b}^{E}\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3+\left(I_{x_bx_b}^{G}-I_{z_bz_b}^{G}\right)\omega_1\omega_3+I_{x_bz_b}^{G}\left(\omega_1^2-\omega_3^2\right)+\left(I_{x_bx_b}^{E}-I_{z_bz_b}^{E}\right)\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3$$
$$=I_{y_by_b}^{G}\dot{\omega}_2+\left(I_{x_bx_b}^{G}-I_{z_bz_b}^{G}\right)\omega_1\omega_3+I_{x_bz_b}^{G}\left(\omega_1^2-\omega_3^2\right)+I_{x_bx_b}^{E}\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3$$
$$\Rightarrow\boxed{I_{y_by_b}^{G}\dot{\omega}_2+\left(I_{x_bx_b}^{G}-I_{z_bz_b}^{G}\right)\omega_1\omega_3+I_{x_bz_b}^{G}\left(\omega_1^2-\omega_3^2\right)+I_{x_bx_b}^{E}\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3=\underset{\sim}{T}_{\text{ext}}^{G}\cdot\underset{\sim}{b}_2}$$

$k=3$: Advantage is taken of the assumption that $I_{y_by_b}^{E}=I_{z_bz_b}^{E}$.

$$I_{z_bz_b}^{G}\dot{\omega}_3-I_{x_bz_b}^{G}\dot{\omega}_1-I_{z_bz_b}^{E}\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2+\left(I_{y_by_b}^{G}-I_{x_bx_b}^{G}\right)\omega_1\omega_2+I_{x_bz_b}^{G}\omega_2\omega_3+\left(I_{y_by_b}^{E}-I_{x_bx_b}^{E}\right)\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2$$
$$=I_{z_bz_b}^{G}\dot{\omega}_3-I_{x_bz_b}^{G}\dot{\omega}_1+\left(I_{y_by_b}^{G}-I_{x_bx_b}^{G}\right)\omega_1\omega_2+I_{x_bz_b}^{G}\omega_2\omega_3-I_{x_bx_b}^{E}\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2$$
$$\Rightarrow\boxed{I_{z_bz_b}^{G}\dot{\omega}_3-I_{x_bz_b}^{G}\dot{\omega}_1+\left(I_{y_by_b}^{G}-I_{x_bx_b}^{G}\right)\omega_1\omega_2+I_{x_bz_b}^{G}\omega_2\omega_3-I_{x_bx_b}^{E}\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2=\underset{\sim}{T}_{\text{ext}}^{G}\cdot\underset{\sim}{b}_3}$$

These are ***identical*** to the moment equations found using the ***Newton/Euler equations*** in Unit 2 and ***Lagrange's equations*** in Unit 4 of this volume.

Equation Summary:

$$m_T\left[\left(\dot{u}+\omega_2 w-\omega_3 v\right)\underset{\sim}{b}_1+\left(\dot{v}+\omega_3 u-\omega_1 w\right)\underset{\sim}{b}_2+\left(\dot{w}+\omega_1 v-\omega_2 u\right)\underset{\sim}{b}_3\right].\underset{\sim}{b}_k=\underset{\sim}{F}_{\text{ext}}\cdot\underset{\sim}{b}_k \qquad (k=1,2,3)$$

$$I_{x_bx_b}^G\dot{\omega}_1-I_{x_bz_b}^G\dot{\omega}_3+I_{x_bx_b}^E\dot{\omega}_{E_1}+I_{x_bx_b}^E\dot{\omega}_{E_2}+\left(I_{z_bz_b}^G-I_{y_by_b}^G\right)\omega_2\omega_3-I_{x_bz_b}^G\omega_1\omega_2=\underset{\sim}{T}_{\text{ext}}^G\cdot\underset{\sim}{b}_1$$

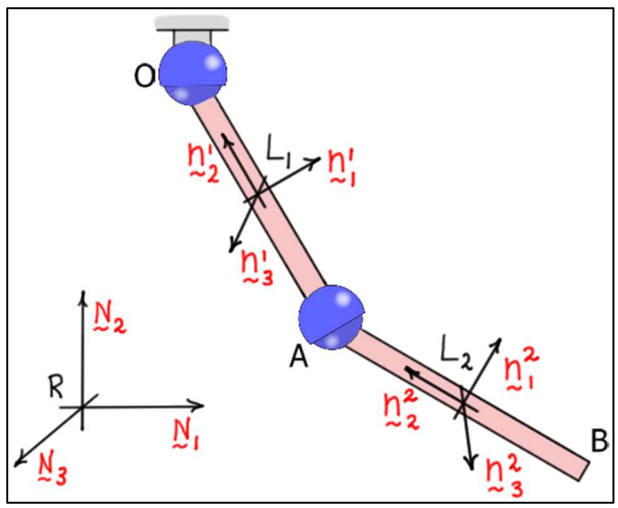
$$I_{y_by_b}^G\dot{\omega}_2+\left(I_{x_bx_b}^G-I_{z_bz_b}^G\right)\omega_1\omega_3+I_{x_bz_b}^G\left(\omega_1^2-\omega_3^2\right)+I_{x_bx_b}^E\left(\omega_{E_1}+\omega_{E_2}\right)\omega_3=\underset{\sim}{T}_{\text{ext}}^G\cdot\underset{\sim}{b}_2$$

$$I_{z_bz_b}^G\dot{\omega}_3-I_{x_bz_b}^G\dot{\omega}_1+\left(I_{y_by_b}^G-I_{x_bx_b}^G\right)\omega_1\omega_2+I_{x_bz_b}^G\omega_2\omega_3-I_{x_bx_b}^E\left(\omega_{E_1}+\omega_{E_2}\right)\omega_2=\underset{\sim}{T}_{\text{ext}}^G\cdot\underset{\sim}{b}_3$$

These represent a set of ***six first-order, ordinary differential equations*** in the ***six generalized speeds*** $(u,v,w,\omega_1,\omega_2,\omega_3)$. These equations must be ***supplemented*** with ***kinematical differential equations*** to track changes in the ***position*** and ***orientation*** of the aircraft. As in Example 2, changes in ***orientation*** can be tracked with a set of ***three orientation angles*** or with ***a set of four Euler parameters***.

Example 6: Double Pendulum or Arm

The system shown is a ***three-dimensional double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ***ball and socket*** joints at O and A. The ***orientation*** of each link is defined relative to the ground using a 3-1-3 ***body-fixed*** rotation sequence. The ***lengths*** of the links are ℓ_1 and ℓ_2. The links are assumed to be ***slender bars*** with mass centers at their ***midpoints***.



Reference frames:

$R:\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3$ (fixed frame)

$L_i:\underset{\sim}{n}_1^i,\underset{\sim}{n}_2^i,\underset{\sim}{n}_3^i\ (i=1,2)$ (fixed in the two links)

Find:

Using Kane's equations, find the ***equations of motion*** describing the ***free motion*** of the double pendulum under the action of ***gravity***. Assume the $\underset{\sim}{N}_2$ direction is ***vertical***. Use the six independent, body-fixed angular velocity components $\{u_1,u_2,u_3,u_4,u_5,u_6\}=\{\omega_{11},\omega_{12},\omega_{13},\omega_{21},\omega_{22},\omega_{23}\}$ as the six independent generalized speeds where

$${}^R\underset{\sim}{\omega}_{L_1}=\omega_{11}\,\underset{\sim}{n}_1^1+\omega_{12}\,\underset{\sim}{n}_2^1+\omega_{13}\,\underset{\sim}{n}_3^1 \qquad {}^R\underset{\sim}{\omega}_{L_2}=\omega_{21}\,\underset{\sim}{n}_1^2+\omega_{22}\,\underset{\sim}{n}_2^2+\omega_{23}\,\underset{\sim}{n}_3^2$$

Solution:

Previous Results

The ***transformation matrices*** and the ***body-fixed components*** of the ***angular velocities*** of the links $L_i\ (i=1,2)$ are as given in Unit 5 of Volume I.

$$\boxed{\left[R_i\right]=\begin{bmatrix} C_{i1}C_{i3}-S_{i1}C_{i2}S_{i3} & S_{i1}C_{i3}+C_{i1}C_{i2}S_{i3} & S_{i2}S_{i3} \\ -C_{i1}S_{i3}-S_{i1}C_{i2}C_{i3} & -S_{i1}S_{i3}+C_{i1}C_{i2}C_{i3} & S_{i2}C_{i3} \\ S_{i1}S_{i2} & -C_{i1}S_{i2} & C_{i2}\end{bmatrix}}\quad \boxed{\begin{aligned}\omega_{i1}&=\dot\theta_{i1}S_{i2}S_{i3}+\dot\theta_{i2}C_{i3}\\ \omega_{i2}&=\dot\theta_{i1}S_{i2}C_{i3}-\dot\theta_{i2}S_{i3}\\ \omega_{i3}&=\dot\theta_{i3}+\dot\theta_{i1}C_{i2}\end{aligned}}\quad (i=1,2)$$

Angular Momentum:

In the analysis that follows, the ***inertia matrices*** of the links about their mass centers are written as

$$\boxed{\left[I_{G_i}\right]_B=\begin{bmatrix} I_{11}^i & 0 & 0\\ 0 & I_{22}^i & 0\\ 0 & 0 & I_{33}^i\end{bmatrix}}\quad (i=1,2)$$

Using these inertia matrices, the ***angular momenta*** of the links about their mass centers can be written as

$$\underset{\sim}{H}_{G_i}=\underset{\approx}{I}_{G_i}\cdot {}^R\underset{\sim}{\omega}_{L_i}\ \rightarrow\ \begin{bmatrix} I_{11}^i & 0 & 0\\ 0 & I_{22}^i & 0\\ 0 & 0 & I_{33}^i\end{bmatrix}\begin{Bmatrix}\omega_{i1}\\ \omega_{i2}\\ \omega_{i3}\end{Bmatrix}\ \Rightarrow \boxed{\underset{\sim}{H}_{G_i}=I_{11}^i\omega_{i1}\,\underset{\sim}{n}_1^i+I_{22}^i\omega_{i2}\,\underset{\sim}{n}_2^i+I_{33}^i\omega_{i3}\,\underset{\sim}{n}_3^i}\quad (i=1,2)$$

Kinematics:

The ***velocities*** of the mass centers of the links can be written as

$$\boxed{{}^R\underset{\sim}{v}_{G_1}={}^R\underset{\sim}{v}_{G_1/O}={}^R\underset{\sim}{\omega}_{L_1}\times\underset{\sim}{r}_{G_1/O}=\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \omega_{11} & \omega_{12} & \omega_{13}\\ 0 & -\frac{1}{2}\ell_1 & 0\end{vmatrix}=\tfrac{1}{2}\ell_1\left(\omega_{13}\,\underset{\sim}{n}_1^1-\omega_{11}\,\underset{\sim}{n}_3^1\right)}$$

$${}^R\underset{\sim}{v}_{G_2}={}^R\underset{\sim}{v}_{A}+{}^R\underset{\sim}{v}_{G_2/A}=2\,{}^R\underset{\sim}{v}_{G_1}+{}^R\underset{\sim}{\omega}_{L_2}\times\underset{\sim}{r}_{G_2/A}=\ell_1\left(\omega_{13}\,\underset{\sim}{n}_1^1-\omega_{11}\,\underset{\sim}{n}_3^1\right)+\begin{vmatrix}\underset{\sim}{n}_1^2 & \underset{\sim}{n}_2^2 & \underset{\sim}{n}_3^2\\ \omega_{21} & \omega_{22} & \omega_{23}\\ 0 & -\frac{1}{2}\ell_2 & 0\end{vmatrix}$$

$$\Rightarrow\boxed{{}^R\underset{\sim}{v}_{G_2}=\ell_1\left(\omega_{13}\,\underset{\sim}{n}_1^1-\omega_{11}\,\underset{\sim}{n}_3^1\right)+\tfrac{1}{2}\ell_2\left(\omega_{23}\,\underset{\sim}{n}_1^2-\omega_{21}\,\underset{\sim}{n}_3^2\right)}$$

The ***accelerations*** of the mass centers of the links can be calculated similarly.

$${}^R\underset{\sim}{a}_{G_1}={}^R\underset{\sim}{a}_{G_1/O}=\left({}^R\underset{\sim}{\alpha}_{L_1}\times\underset{\sim}{r}_{G_1/O}\right)+{}^R\underset{\sim}{\omega}_{L_1}\times\left({}^R\underset{\sim}{\omega}_{L_1}\times{}^R\underset{\sim}{r}_{G_1/O}\right)$$

$$=\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \dot\omega_{11} & \dot\omega_{12} & \dot\omega_{13}\\ 0 & -\frac{1}{2}\ell_1 & 0\end{vmatrix}+{}^R\underset{\sim}{\omega}_{L_1}\times\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \omega_{11} & \omega_{12} & \omega_{13}\\ 0 & -\frac{1}{2}\ell_1 & 0\end{vmatrix}=\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \dot\omega_{11} & \dot\omega_{12} & \dot\omega_{13}\\ 0 & -\frac{1}{2}\ell_1 & 0\end{vmatrix}+\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \omega_{11} & \omega_{12} & \omega_{13}\\ \frac{1}{2}\ell_1\omega_{13} & 0 & -\frac{1}{2}\ell_1\omega_{11}\end{vmatrix}$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{a}_{G_1} = \tfrac{1}{2}\ell_1\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}^1_1 + \tfrac{1}{2}\ell_1\left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}^1_2 - \tfrac{1}{2}\ell_1\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}^1_3}$$

$$ {}^R\underset{\sim}{a}_{G_2} = {}^R\underset{\sim}{a}_{A} + {}^R\underset{\sim}{a}_{G_2/A} = 2\,{}^R\underset{\sim}{a}_{G_1} + \left({}^R\underset{\sim}{\alpha}_{L_1}\times \underset{\sim}{r}_{G_1/O}\right) + {}^R\underset{\sim}{\omega}_{L_1}\times\left({}^R\underset{\sim}{\omega}_{L_1}\times {}^R\underset{\sim}{r}_{G_1/O}\right)$$

$$= \ell_1\left[\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}^1_1 + \left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}^1_2 - \left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}^1_3\right] + \begin{vmatrix} \underset{\sim}{n}^2_1 & \underset{\sim}{n}^2_2 & \underset{\sim}{n}^2_3 \\ \dot{\omega}_{21} & \dot{\omega}_{22} & \dot{\omega}_{23} \\ 0 & -\frac{1}{2}\ell_2 & 0 \end{vmatrix}$$

$$+ {}^R\underset{\sim}{\omega}_{L_2}\times \begin{vmatrix} \underset{\sim}{n}^2_1 & \underset{\sim}{n}^2_2 & \underset{\sim}{n}^2_3 \\ \omega_{21} & \omega_{22} & \omega_{23} \\ 0 & -\frac{1}{2}\ell_2 & 0 \end{vmatrix}$$

$$= \ell_1\left[\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}^1_1 + \left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}^1_2 - \left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}^1_3\right]$$

$$+ \tfrac{1}{2}\ell_2\left(\dot{\omega}_{23}\,\underset{\sim}{n}^2_1 - \dot{\omega}_{21}\,\underset{\sim}{n}^2_3\right) + \begin{vmatrix} \underset{\sim}{n}^2_1 & \underset{\sim}{n}^2_2 & \underset{\sim}{n}^2_3 \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \frac{1}{2}\ell_2\omega_{23} & 0 & -\frac{1}{2}\ell_2\omega_{21} \end{vmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} {}^R\underset{\sim}{a}_{G_2} = {} & \ell_1\left[\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}^1_1 + \left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}^1_2 - \left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}^1_3\right] \\ & + \tfrac{1}{2}\ell_2\left[\left(\dot{\omega}_{23}-\omega_{21}\omega_{22}\right)\underset{\sim}{n}^2_1 + \left(\omega_{23}^2+\omega_{21}^2\right)\underset{\sim}{n}^2_2 - \left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)\underset{\sim}{n}^2_3\right] \end{aligned}}$$

Partial Angular Velocities and Partial Velocities:

The ***partial angular velocities*** of the links can be written as follows.

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \omega_{1k}} = \underset{\sim}{n}^1_k} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_{L_2}}{\partial \omega_{2k}} = \underset{\sim}{n}^2_k} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \omega_{2k}} = \underset{\sim}{0}} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_{L_2}}{\partial \omega_{1k}} = \underset{\sim}{0}} \quad \left(k=1,2,3\right)$$

The ***partial velocities*** of the mass centers of the two links can be written as follows.

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \omega_{11}} = -\tfrac{1}{2}\ell_1\,\underset{\sim}{n}^1_3} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \omega_{12}} = \underset{\sim}{0}} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \omega_{13}} = \tfrac{1}{2}\ell_1\,\underset{\sim}{n}^1_1} \quad \text{and} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial \omega_{2k}} = \underset{\sim}{0}} \quad \left(k=1,2,3\right)$$

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \omega_{11}} = -\ell_1\,\underset{\sim}{n}^1_3} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \omega_{12}} = \underset{\sim}{0}} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \omega_{13}} = \ell_1\,\underset{\sim}{n}^1_1}$$

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \omega_{21}} = -\tfrac{1}{2}\ell_2\,\underset{\sim}{n}^2_3} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \omega_{22}} = \underset{\sim}{0}} \quad \boxed{\frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial \omega_{23}} = \tfrac{1}{2}\ell_2\,\underset{\sim}{n}^2_1}$$

Generalized Forces:

The only ***active forces*** in this system are the ***two weight forces*** that act at the mass centers of the links. The generalized forces associated with these two forces can be written as

$$\boxed{F_{u_k} = \underset{\sim}{W}_1 \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_k} + \underset{\sim}{W}_2 \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial u_k}} \quad (k=1,\ldots,6)$$

Specifically,

$$\boxed{F_{u_1} = F_{\omega_{11}} = -m_1 g\, \underset{\sim}{N}_2 \cdot \left(-\tfrac{1}{2}\ell_1 \underset{\sim}{n}_3^1\right) - m_2 g\, \underset{\sim}{N}_2 \cdot \left(-\ell_1 \underset{\sim}{n}_3^1\right) = \left(\tfrac{1}{2}m_1 + m_2\right) g \ell_1 \left(\underset{\sim}{N}_2 \cdot \underset{\sim}{n}_3^1\right) = \left(\tfrac{1}{2}m_1 + m_2\right) g \ell_1 R_{32}^1}$$

$$\boxed{F_{u_2} = F_{\omega_{12}} = -m_1 g\, \underset{\sim}{N}_2 \cdot \left(\underset{\sim}{0}\right) - m_2 g\, \underset{\sim}{N}_2 \cdot \left(\underset{\sim}{0}\right) = 0}$$

$$\boxed{F_{u_3} = F_{\omega_{13}} = -m_1 g\, \underset{\sim}{N}_2 \cdot \left(\tfrac{1}{2}\ell_1 \underset{\sim}{n}_1^1\right) - m_2 g\, \underset{\sim}{N}_2 \cdot \left(\ell_1 \underset{\sim}{n}_1^1\right) = -\left(\tfrac{1}{2}m_1 + m_2\right) g \ell_1 \left(\underset{\sim}{N}_2 \cdot \underset{\sim}{n}_1^1\right) = -\left(\tfrac{1}{2}m_1 + m_2\right) g \ell_1 R_{12}^1}$$

$$\boxed{F_{u_4} = F_{\omega_{21}} = -m_1 g\, \underset{\sim}{N}_2 \cdot \left(\underset{\sim}{0}\right) - m_2 g\, \underset{\sim}{N}_2 \cdot \left(-\tfrac{1}{2}\ell_2 \underset{\sim}{n}_3^2\right) = \tfrac{1}{2} m_2 g \ell_2 \left(\underset{\sim}{N}_2 \cdot \underset{\sim}{n}_3^2\right) = \tfrac{1}{2} m_2 g \ell_2 R_{32}^2}$$

$$\boxed{F_{u_5} = F_{\omega_{22}} = -m_1 g\, \underset{\sim}{N}_2 \cdot \left(\underset{\sim}{0}\right) - m_2 g\, \underset{\sim}{N}_2 \cdot \left(\underset{\sim}{0}\right) = 0}$$

$$\boxed{F_{u_6} = F_{\omega_{23}} = -m_1 g\, \underset{\sim}{N}_2 \cdot \left(\underset{\sim}{0}\right) - m_2 g\, \underset{\sim}{N}_2 \cdot \left(\tfrac{1}{2}\ell_2 \underset{\sim}{n}_1^2\right) = -\tfrac{1}{2} m_2 g \ell_2 \left(\underset{\sim}{N}_2 \cdot \underset{\sim}{n}_1^2\right) = -\tfrac{1}{2} m_2 g \ell_2 R_{12}^2}$$

Here, R_{32}^i and R_{12}^i $(i=1,2)$ are entries of the transformation matrices $\left[R_i\right]$ $(i=1,2)$.

Kane's Equations:

Kane's equations for the two link pendulum can be written as

$$\boxed{\sum_{i=1}^{2}\left(m_i\, {}^R\underset{\sim}{a}_{G_i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k}\right) + \sum_{i=1}^{2}\left[\left(\underset{\approx}{I}_{G_i} \cdot {}^R\underset{\sim}{\alpha}_{L_i}\right) + \left({}^R\underset{\sim}{\omega}_{L_i} \times \underset{\sim}{H}_{G_i}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_i}}{\partial u_k} = F_{u_k}} \quad (k=1,\ldots,6)$$

The calculation of each of the terms on the ***left side*** of Kane's equations follows. Advantage is taken of the transformation matrices to transform components into the same reference frames.

1. $$m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_1} = m_1\left[\tfrac{1}{2}\ell_1\left(\dot{\omega}_{13} - \omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1 + \tfrac{1}{2}\ell_1\left(\omega_{13}^2 + \omega_{11}^2\right)\underset{\sim}{n}_2^1 - \tfrac{1}{2}\ell_1\left(\dot{\omega}_{11} + \omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\right] \cdot \left[-\tfrac{1}{2}\ell_1 \underset{\sim}{n}_3^1\right]$$

$$\boxed{m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_1} = \tfrac{1}{4} m_1 \ell_1^2 \left(\dot{\omega}_{11} + \omega_{12}\omega_{13}\right)}$$

$$m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_3} = m_1\left[\tfrac{1}{2}\ell_1\left(\dot{\omega}_{13} - \omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1 + \tfrac{1}{2}\ell_1\left(\omega_{13}^2 + \omega_{11}^2\right)\underset{\sim}{n}_2^1 - \tfrac{1}{2}\ell_1\left(\dot{\omega}_{11} + \omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\right] \cdot \left[\tfrac{1}{2}\ell_1 \underset{\sim}{n}_1^1\right]$$

$$\boxed{m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_3} = \tfrac{1}{4} m_1 \ell_1^2 \left(\dot{\omega}_{13} - \omega_{11}\omega_{12}\right)}$$

$$\boxed{m_1\, {}^R\underset{\sim}{a}_{G_1} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_1}}{\partial u_k} = 0} \quad \left(k = 2,4,5,6\right)$$

2. $$m_2\, {}^R\underset{\sim}{a}_{G_2} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_2}}{\partial u_1} = m_2\left[\ell_1\left[\left(\dot{\omega}_{13} - \omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1 + \left(\omega_{13}^2 + \omega_{11}^2\right)\underset{\sim}{n}_2^1 - \left(\dot{\omega}_{11} + \omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\right]\right] \cdot \left[-\ell_1 \underset{\sim}{n}_3^1\right]$$

$$+m_2\left[\tfrac{1}{2}\ell_2\left[\left(\dot{\omega}_{23} - \omega_{21}\omega_{22}\right)\underset{\sim}{n}_1^2 + \left(\omega_{23}^2 + \omega_{21}^2\right)\underset{\sim}{n}_2^2 - \left(\dot{\omega}_{21} + \omega_{22}\omega_{23}\right)\underset{\sim}{n}_3^2\right]\right] \cdot \left[-\ell_1 \underset{\sim}{n}_3^1\right]$$

$$\Rightarrow \boxed{m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_1}=m_2\ell_1^2\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)-\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix}R_{31}^1 & R_{32}^1 & R_{33}^1\end{bmatrix}\begin{bmatrix}R_2\end{bmatrix}^T\begin{Bmatrix}\left(\dot{\omega}_{23}-\omega_{21}\omega_{22}\right)\\\left(\omega_{23}^2+\omega_{21}^2\right)\\-\left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)\end{Bmatrix}}$$

$$\boxed{m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_2}=m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\underset{\sim}{0}=0}$$

$$m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_3}=m_2\left[\ell_1\left[\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1+\left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}_2^1-\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\right]\right]\cdot\left[\ell_1\,\underset{\sim}{n}_1^1\right]$$
$$+m_2\left[\tfrac{1}{2}\ell_2\left[\left(\dot{\omega}_{23}-\omega_{21}\omega_{22}\right)\underset{\sim}{n}_1^2+\left(\omega_{23}^2+\omega_{21}^2\right)\underset{\sim}{n}_2^2-\left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)\underset{\sim}{n}_3^2\right]\right]\cdot\left[\ell_1\,\underset{\sim}{n}_1^1\right]$$

$$\Rightarrow \boxed{m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_3}=m_2\ell_1^2\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)+\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix}R_{11}^1 & R_{12}^1 & R_{13}^1\end{bmatrix}\begin{bmatrix}R_2\end{bmatrix}^T\begin{Bmatrix}\left(\dot{\omega}_{23}-\omega_{21}\omega_{22}\right)\\\left(\omega_{23}^2+\omega_{21}^2\right)\\-\left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)\end{Bmatrix}}$$

$$m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_4}=m_2\left[\ell_1\left[\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1+\left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}_2^1-\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\right]\right]\cdot\left[-\tfrac{1}{2}\ell_2\,\underset{\sim}{n}_3^2\right]$$
$$+m_2\left[\tfrac{1}{2}\ell_2\left[\left(\dot{\omega}_{23}-\omega_{21}\omega_{22}\right)\underset{\sim}{n}_1^2+\left(\omega_{23}^2+\omega_{21}^2\right)\underset{\sim}{n}_2^2-\left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)\underset{\sim}{n}_3^2\right]\right]\cdot\left[-\tfrac{1}{2}\ell_2\,\underset{\sim}{n}_3^2\right]$$

$$\boxed{m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_4}=-\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix}R_{31}^2 & R_{32}^2 & R_{33}^2\end{bmatrix}\begin{bmatrix}R_1\end{bmatrix}^T\begin{Bmatrix}\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\\\left(\omega_{13}^2+\omega_{11}^2\right)\\-\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\end{Bmatrix}+\tfrac{1}{4}m_2\ell_2^2\left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)}$$

$$\boxed{m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_5}=m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\underset{\sim}{0}=0}$$

$$m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_6}=m_2\left[\ell_1\left[\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1+\left(\omega_{13}^2+\omega_{11}^2\right)\underset{\sim}{n}_2^1-\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\right]\right]\cdot\left[\tfrac{1}{2}\ell_2\,\underset{\sim}{n}_1^2\right]$$
$$+m_2\left[\tfrac{1}{2}\ell_2\left[\left(\dot{\omega}_{23}-\omega_{21}\omega_{22}\right)\underset{\sim}{n}_1^2+\left(\omega_{23}^2+\omega_{21}^2\right)\underset{\sim}{n}_2^2-\left(\dot{\omega}_{21}+\omega_{22}\omega_{23}\right)\underset{\sim}{n}_3^2\right]\right]\cdot\left[\tfrac{1}{2}\ell_2\,\underset{\sim}{n}_1^2\right]$$

$$m_2\,{}^R\underset{\sim}{a}_{G_2}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_{G_2}}{\partial u_6}=\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix}R_{11}^2 & R_{12}^2 & R_{13}^2\end{bmatrix}\begin{bmatrix}R_1\end{bmatrix}^T\begin{Bmatrix}(\dot{\omega}_{13}-\omega_{11}\omega_{12})\\(\omega_{13}^2+\omega_{11}^2)\\-(\dot{\omega}_{11}+\omega_{12}\omega_{13})\end{Bmatrix}+\tfrac{1}{4}m_2\ell_2^2(\dot{\omega}_{23}-\omega_{21}\omega_{22})$$

3. $$\left(\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\alpha}_{L_1}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_1}}{\partial u_k}=\left(I_{11}^1\dot{\omega}_{11}\,\underset{\sim}{n}_1^1+I_{22}^1\dot{\omega}_{12}\,\underset{\sim}{n}_2^1+I_{33}^1\dot{\omega}_{13}\,\underset{\sim}{n}_3^1\right)\cdot\underset{\sim}{n}_k^1=I_{kk}^1\,\dot{\omega}_{1k}\qquad(k=1,2,3)$$

$$\left(\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\alpha}_{L_1}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_1}}{\partial u_k}=\left(\underset{\approx}{I}_{G_1}\cdot{}^R\underset{\sim}{\alpha}_{L_1}\right)\cdot\underset{\sim}{0}=0\qquad(k=4,5,6)$$

4. $$\left(\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\alpha}_{L_2}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_2}}{\partial u_k}=\left(\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\alpha}_{L_2}\right)\cdot\underset{\sim}{0}=0\qquad(k=1,2,3)$$

$$\left(\underset{\approx}{I}_{G_2}\cdot{}^R\underset{\sim}{\alpha}_{L_2}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_2}}{\partial u_{k+3}}=\left(I_{11}^2\dot{\omega}_{21}\,\underset{\sim}{n}_1^2+I_{22}^2\dot{\omega}_{22}\,\underset{\sim}{n}_2^2+I_{33}^2\dot{\omega}_{23}\,\underset{\sim}{n}_3^2\right)\cdot\underset{\sim}{n}_k^2=I_{kk}^2\dot{\omega}_{2k}\qquad(k=1,2,3)$$

5. $$\begin{aligned}{}^R\underset{\sim}{\omega}_{L_i}\times\underset{\sim}{H}_{G_i}&=\begin{vmatrix}\underset{\sim}{n}_1^i & \underset{\sim}{n}_2^i & \underset{\sim}{n}_3^i\\ \omega_{i1} & \omega_{i2} & \omega_{i3}\\ I_{11}^i\omega_{i1} & I_{22}^i\omega_{i2} & I_{33}^i\omega_{i3}\end{vmatrix}\\&=\left(I_{33}^i-I_{22}^i\right)\omega_{i2}\omega_{i3}\,\underset{\sim}{n}_1^i+\left(I_{11}^i-I_{33}^i\right)\omega_{i1}\omega_{i3}\,\underset{\sim}{n}_2^i+\left(I_{22}^i-I_{11}^i\right)\omega_{i1}\omega_{i2}\,\underset{\sim}{n}_3^i\end{aligned}$$

6. $$\begin{aligned}\left({}^R\underset{\sim}{\omega}_{L_1}\times\underset{\sim}{H}_{G_1}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_1}}{\partial u_1}&=\left[\left(I_{33}^1-I_{22}^1\right)\omega_{12}\omega_{13}\,\underset{\sim}{n}_1^1+\left(I_{11}^1-I_{33}^1\right)\omega_{11}\omega_{13}\,\underset{\sim}{n}_2^1+\left(I_{22}^1-I_{11}^1\right)\omega_{11}\omega_{12}\,\underset{\sim}{n}_3^1\right]\cdot\underset{\sim}{n}_1^1\\&=\left(I_{33}^1-I_{22}^1\right)\omega_{12}\omega_{13}\end{aligned}$$

$$\begin{aligned}\left({}^R\underset{\sim}{\omega}_{L_1}\times\underset{\sim}{H}_{G_1}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_1}}{\partial u_2}&=\left[\left(I_{33}^1-I_{22}^1\right)\omega_{12}\omega_{13}\,\underset{\sim}{n}_1^1+\left(I_{11}^1-I_{33}^1\right)\omega_{11}\omega_{13}\,\underset{\sim}{n}_2^1+\left(I_{22}^1-I_{11}^1\right)\omega_{11}\omega_{12}\,\underset{\sim}{n}_3^1\right]\cdot\underset{\sim}{n}_2^1\\&=\left(I_{11}^1-I_{33}^1\right)\omega_{11}\omega_{13}\end{aligned}$$

$$\begin{aligned}\left({}^R\underset{\sim}{\omega}_{L_1}\times\underset{\sim}{H}_{G_1}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_1}}{\partial u_3}&=\left[\left(I_{33}^1-I_{22}^1\right)\omega_{12}\omega_{13}\,\underset{\sim}{n}_1^1+\left(I_{11}^1-I_{33}^1\right)\omega_{11}\omega_{13}\,\underset{\sim}{n}_2^1+\left(I_{22}^1-I_{11}^1\right)\omega_{11}\omega_{12}\,\underset{\sim}{n}_3^1\right]\cdot\underset{\sim}{n}_3^1\\&=\left(I_{22}^1-I_{11}^1\right)\omega_{11}\omega_{12}\end{aligned}$$

$$\left({}^R\underset{\sim}{\omega}_{L_1}\times\underset{\sim}{H}_{G_1}\right)\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_{L_1}}{\partial u_k}=\left({}^R\underset{\sim}{\omega}_{L_1}\times\underset{\sim}{H}_{G_1}\right)\cdot\underset{\sim}{0}=0\qquad k=4,5,6$$

7. $$\left({}^{R}\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{H}_{G_2}\right) \cdot \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{L_2}}{\partial u_k} = \left({}^{R}\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{H}_{G_2}\right) \cdot \underset{\sim}{0} = 0 \qquad k = 1,2,3$$

$$\begin{aligned}\left({}^{R}\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{H}_{G_2}\right) \cdot \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{L_2}}{\partial u_4} &= \left[\left(I_{33}^2 - I_{22}^2\right)\omega_{22}\omega_{23}\, \underset{\sim}{n}_1^2 + \left(I_{11}^2 - I_{33}^2\right)\omega_{21}\omega_{23}\, \underset{\sim}{n}_2^2 + \left(I_{22}^2 - I_{11}^2\right)\omega_{21}\omega_{22}\, \underset{\sim}{n}_3^2\right] \cdot \underset{\sim}{n}_1^2 \\ &= \left(I_{33}^2 - I_{22}^2\right)\omega_{22}\omega_{23}\end{aligned}$$

$$\begin{aligned}\left({}^{R}\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{H}_{G_2}\right) \cdot \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{L_2}}{\partial u_5} &= \left[\left(I_{33}^2 - I_{22}^2\right)\omega_{22}\omega_{23}\, \underset{\sim}{n}_1^2 + \left(I_{11}^2 - I_{33}^2\right)\omega_{21}\omega_{23}\, \underset{\sim}{n}_2^2 + \left(I_{22}^2 - I_{11}^2\right)\omega_{21}\omega_{22}\, \underset{\sim}{n}_3^2\right] \cdot \underset{\sim}{n}_2^2 \\ &= \left(I_{11}^2 - I_{33}^2\right)\omega_{21}\omega_{23}\end{aligned}$$

$$\begin{aligned}\left({}^{R}\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{H}_{G_2}\right) \cdot \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{L_2}}{\partial u_6} &= \left[\left(I_{33}^2 - I_{22}^2\right)\omega_{22}\omega_{23}\, \underset{\sim}{n}_1^2 + \left(I_{11}^2 - I_{33}^2\right)\omega_{21}\omega_{23}\, \underset{\sim}{n}_2^2 + \left(I_{22}^2 - I_{11}^2\right)\omega_{21}\omega_{22}\, \underset{\sim}{n}_3^2\right] \cdot \underset{\sim}{n}_3^2 \\ &= \left(I_{22}^2 - I_{11}^2\right)\omega_{21}\omega_{22}\end{aligned}$$

Substituting these results into Kane's equations gives the following ***six equations of motion***.

$$\begin{aligned}&\left(I_{11}^1 + \tfrac{1}{4}m_1\ell_1^2 + m_2\ell_1^2\right)\dot{\omega}_{11} + \left(I_{33}^1 + \tfrac{1}{4}m_1\ell_1^2 - I_{22}^1 + m_2\ell_1^2\right)\omega_{12}\omega_{13} \\ &-\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix} R_{31}^1 & R_{32}^1 & R_{33}^1\end{bmatrix}\begin{bmatrix} R_2\end{bmatrix}^T \begin{Bmatrix} \left(\dot{\omega}_{23} - \omega_{21}\omega_{22}\right) \\ \left(\omega_{23}^2 + \omega_{21}^2\right) \\ -\left(\dot{\omega}_{21} + \omega_{22}\omega_{23}\right)\end{Bmatrix} = \left(\tfrac{1}{2}m_1 + m_2\right) g\ell_1 R_{32}^1\end{aligned}$$

$$I_{22}^1\, \dot{\omega}_{12} + \left(I_{11}^1 - I_{33}^1\right)\omega_{11}\omega_{13} = 0$$

$$\begin{aligned}&\left(I_{33}^1 + \tfrac{1}{4}m_1\ell_1^2 + m_2\ell_1^2\right)\dot{\omega}_{13} + \left(I_{22}^1 - I_{11}^1 - \tfrac{1}{4}m_1\ell_1^2 - m_2\ell_1^2\right)\omega_{11}\omega_{12} \\ &+\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix} R_{11}^1 & R_{12}^1 & R_{13}^1\end{bmatrix}\begin{bmatrix} R_2\end{bmatrix}^T \begin{Bmatrix} \left(\dot{\omega}_{23} - \omega_{21}\omega_{22}\right) \\ \left(\omega_{23}^2 + \omega_{21}^2\right) \\ -\left(\dot{\omega}_{21} + \omega_{22}\omega_{23}\right)\end{Bmatrix} = -\left(\tfrac{1}{2}m_1 + m_2\right) g\ell_1 R_{12}^1\end{aligned}$$

$$\begin{aligned}&\left(I_{11}^2 + \tfrac{1}{4}m_2\ell_2^2\right)\dot{\omega}_{21} + \left(I_{33}^2 + \tfrac{1}{4}m_2\ell_2^2 - I_{22}^2\right)\omega_{22}\omega_{23} \\ &-\tfrac{1}{2}m_2\ell_1\ell_2\begin{bmatrix} R_{31}^2 & R_{32}^2 & R_{33}^2\end{bmatrix}\begin{bmatrix} R_1\end{bmatrix}^T \begin{Bmatrix} \left(\dot{\omega}_{13} - \omega_{11}\omega_{12}\right) \\ \left(\omega_{13}^2 + \omega_{11}^2\right) \\ -\left(\dot{\omega}_{11} + \omega_{12}\omega_{13}\right)\end{Bmatrix} = \tfrac{1}{2}m_2 g\ell_2 R_{32}^2\end{aligned}$$

$$I_{22}^2\dot{\omega}_{22} + \left(I_{11}^2 - I_{33}^2\right)\omega_{21}\omega_{23} = 0$$

$$\left(I_{33}^2+\tfrac{1}{4}m_2\ell_2^2\right)\dot{\omega}_{23}+\left(I_{22}^2-I_{11}^2-\tfrac{1}{4}m_2\ell_2^2\right)\omega_{21}\omega_{22}$$
$$+\tfrac{1}{2}m_2\ell_1\ell_2\left[R_{11}^2 \quad R_{12}^2 \quad R_{13}^2\right]\left[R_1\right]^T\begin{Bmatrix}\left(\dot{\omega}_{13}-\omega_{11}\omega_{12}\right)\\ \left(\omega_{13}^2+\omega_{11}^2\right)\\ -\left(\dot{\omega}_{11}+\omega_{12}\omega_{13}\right)\end{Bmatrix}=-\tfrac{1}{2}m_2 g\ell_2 R_{12}^2$$

Notes:

1. If the inertias of the two links about their mass centers for the $\underset{\sim}{n}_1^i$ and $\underset{\sim}{n}_3^i$ directions are equal, that is if $I_{11}^1=I_{33}^1$ and $I_{11}^2=I_{33}^2$, then the second and fifth equations simplify to

$$\boxed{I_{22}^1\,\dot{\omega}_{12}=0} \quad \Rightarrow \boxed{\omega_{12}=\text{constant}} \qquad\qquad \boxed{I_{22}^2\dot{\omega}_{22}=0} \quad \Rightarrow \boxed{\omega_{22}=\text{constant}}$$

2. If the inertias I_{22}^1 and I_{22}^2 are ***assumed*** to be ***zero***, then the second and fifth equations simply state that $0=0$. However, the angular velocity components ω_{12} and ω_{22} appear in the other four equations. This renders the equations ***unsolvable*** as there are only ***four equations*** with ***six unknowns***.
3. To track changes in the ***orientation angles***, Kane's equations are supplemented with the ***six kinematical differential equations*** relating the angular velocity components to the orientation angle derivatives.

$$\begin{aligned}\dot{\theta}_{i1}&=\left(\omega_{i1}S_{i3}+\omega_{i2}C_{i3}\right)\big/S_{i2}\\ \dot{\theta}_{i2}&=\omega_{i1}C_{i3}-\omega_{i2}S_{i3}\\ \dot{\theta}_{i3}&=\omega_{i3}-\left(\omega_{i1}S_{i3}+\omega_{i2}C_{i3}\right)C_{i2}\big/S_{i2}\end{aligned} \qquad (i=1,2)$$

4. Using ***d'Alembert's principle*** or ***Lagrange's equations*** to generate the equations of motion for this system produces ***much more complicated results*** due to the need for choosing a set of ***independent generalized coordinates*** on which to base the equations.
5. The third set of terms on the left side of the first, third, fourth, and sixth equations involve ***matrix-vector products*** associated with the ***transformation matrices*** of the two links. Recall, in this case, the rows of a transformation matrices $\left[R_i\right]$ $(i=1,2)$ (and hence the columns of their transposes) contain the base-frame components of the unit vectors fixed in the links. Hence, the matrix-vector products in the equations of motion can be interpreted geometrically as cosines of the angles between the unit vectors fixed in link L_1 and those fixed in link L_2. Specifically,

$$\boxed{\left[R_{31}^1 \quad R_{32}^1 \quad R_{33}^1\right]\left[R_2\right]^T=\left[\underset{\sim}{n}_3^1\cdot\underset{\sim}{n}_1^2 \quad \underset{\sim}{n}_3^1\cdot\underset{\sim}{n}_2^2 \quad \underset{\sim}{n}_3^1\cdot\underset{\sim}{n}_3^2\right]=\left[\cos\left(\underset{\sim}{n}_3^1,\underset{\sim}{n}_1^2\right) \quad \cos\left(\underset{\sim}{n}_3^1,\underset{\sim}{n}_2^2\right) \quad \cos\left(\underset{\sim}{n}_3^1,\underset{\sim}{n}_3^2\right)\right]}$$
$$\boxed{\left[R_{11}^1 \quad R_{12}^1 \quad R_{13}^1\right]\left[R_2\right]^T=\left[\underset{\sim}{n}_1^1\cdot\underset{\sim}{n}_1^2 \quad \underset{\sim}{n}_1^1\cdot\underset{\sim}{n}_2^2 \quad \underset{\sim}{n}_1^1\cdot\underset{\sim}{n}_3^2\right]=\left[\cos\left(\underset{\sim}{n}_1^1,\underset{\sim}{n}_1^2\right) \quad \cos\left(\underset{\sim}{n}_1^1,\underset{\sim}{n}_2^2\right) \quad \cos\left(\underset{\sim}{n}_1^1,\underset{\sim}{n}_3^2\right)\right]}$$

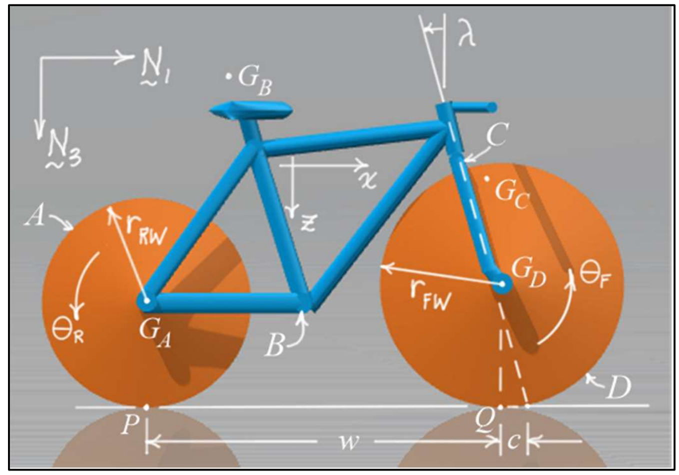
$$\left[R_{31}^2 \quad R_{32}^2 \quad R_{33}^2 \right] \left[R_1 \right]^T = \left[\underset{\sim}{n}_3^2 \cdot \underset{\sim}{n}_1^1 \quad \underset{\sim}{n}_3^2 \cdot \underset{\sim}{n}_2^1 \quad \underset{\sim}{n}_3^2 \cdot \underset{\sim}{n}_3^1 \right] = \left[\cos\left(\underset{\sim}{n}_3^2, \underset{\sim}{n}_1^1\right) \quad \cos\left(\underset{\sim}{n}_3^2, \underset{\sim}{n}_2^1\right) \quad \cos\left(\underset{\sim}{n}_3^2, \underset{\sim}{n}_3^1\right) \right]$$

$$\left[R_{11}^2 \quad R_{12}^2 \quad R_{13}^2 \right] \left[R_1 \right]^T = \left[\underset{\sim}{n}_1^2 \cdot \underset{\sim}{n}_1^1 \quad \underset{\sim}{n}_1^2 \cdot \underset{\sim}{n}_2^1 \quad \underset{\sim}{n}_1^2 \cdot \underset{\sim}{n}_3^1 \right] = \left[\cos\left(\underset{\sim}{n}_1^2, \underset{\sim}{n}_1^1\right) \quad \cos\left(\underset{\sim}{n}_1^2, \underset{\sim}{n}_2^1\right) \quad \cos\left(\underset{\sim}{n}_1^2, \underset{\sim}{n}_3^1\right) \right]$$

Example 7: Free Motion of a Bicycle (for more information, see references 10 (a)-(d))

Configuration: Aligned Position

The analysis that follows is for a ***two-wheeled***, ***upright*** bicycle as shown. The bicycle is modeled as ***four*** interconnected bodies – the rear wheel *A*, the rear frame and rider *B*, the steering column and fork *C*, and the front wheel *D*. The bicycle is assumed to be moving freely on a horizontal surface with the rear wheel contacting the surface at point *P* and the front wheel contacting the surface at point *Q*. The distance between the contact points is the wheelbase *w*.



The steering column is assumed to be tilted relative to the vertical at an angle λ. This angle is a right-hand rotation about the $\underset{\sim}{N}_2 = \underset{\sim}{N}_3 \times \underset{\sim}{N}_1$ direction. The projection of the steering axis onto the horizontal plane is assumed to be a distance *c* in front of the contact point *Q*. Positive rotations of the front and rear wheels are also measured about the $\underset{\sim}{N}_2$ direction, so for forward motion of the bicycle, both angles θ_F and θ_R are ***negative***.

In the configuration shown in the figure above, the *x* and *z* axes fixed in the rear frame *B* are aligned with the global (inertial) directions defined by the unit vector set $R : \left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3 \right)$. The unit vectors $\underset{\sim}{N}_1$ and $\underset{\sim}{N}_2$ are parallel to the horizontal plane, and the unit vector $\underset{\sim}{N}_3$ points vertically downward.

The points G_A, G_B, G_C, and G_D are the mass centers of the bodies and are located using the following data for the bicycle. Recall that body *B* includes both the ***rear frame*** and the ***rider***. All the data given in the table are referenced to the global directions.

w	c	r_{RW}	r_{FW}	$\underset{\sim}{r}_{G_A/P}$	$\underset{\sim}{r}_{G_B/P}$	$\underset{\sim}{r}_{G_C/P}$	$\underset{\sim}{r}_{G_D/P}$	λ
1.02 (m)	0.08 (m)	0.3 (m)	0.35 (m)	$(0, -r_{RW})$	$(0.3, -0.9)$	$(0.9, -0.7)$	$(w, -r_{FW})$	18 (deg)

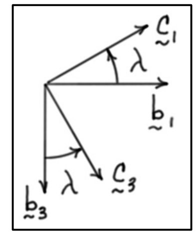
It can be shown using a ***geometric analysis*** that a ***point S*** located on the ***steering axis*** (common to both the rear frame and the steering column) can be located using the following equations.

Horizontal distance from G_A to *S*: $x_S = (w+c)C_\lambda^2 - r_{RW} S_\lambda C_\lambda$

Vertical distance from G_A to *S*: $z_S = x_S \tan(\lambda)$

For convenience in the later analysis, two additional sets of unit vectors are defined. The set $B:\left(\underset{\sim}{b}_1,\underset{\sim}{b}_2,\underset{\sim}{b}_3\right)$ is fixed in the rear frame B, with $\underset{\sim}{b}_1$ pointing in the x-direction (forward), $\underset{\sim}{b}_2$ pointing in the y-direction (to the rider's right), and $\underset{\sim}{b}_3=\underset{\sim}{b}_1\times\underset{\sim}{b}_2$. In the figure above, the unit vectors of B are aligned with the unit vectors of R.

A second set $C:\left(\underset{\sim}{c}_1,\underset{\sim}{c}_2,\underset{\sim}{c}_3\right)$ is fixed in the steering column C. In the position where frame B is aligned with frame R, the unit vectors in frame C are directed as follows. Unit vectors $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_3$ are at an angle λ with their counterparts in the B frame, and $\underset{\sim}{c}_2$ is aligned with $\underset{\sim}{b}_2$. See diagram.



For the analysis that follows, the position vectors of G_B and S relative to G_A are expressed in the B frame, and the position vectors of G_C and G_D relative to S are expressed in the C frame as follows.

$$\underset{\sim}{r}_{G_B/G_A}=x_B\underset{\sim}{b}_1-z_B\underset{\sim}{b}_3 \qquad \underset{\sim}{r}_{S/G_A}=x_S\underset{\sim}{b}_1-z_S\underset{\sim}{b}_3$$

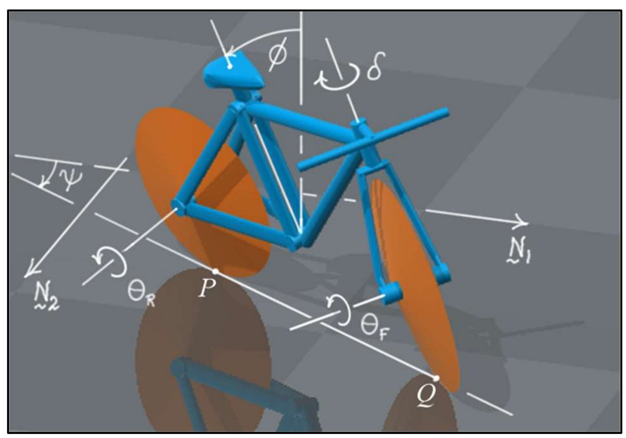
$$\underset{\sim}{r}_{G_C/S}=x_C\underset{\sim}{c}_1+z_C\underset{\sim}{c}_3 \qquad \underset{\sim}{r}_{G_D/S}=x_D\underset{\sim}{c}_1+z_D\underset{\sim}{c}_3$$

The fixed values of the components of these vectors are provided in the adjacent table. The values are based on the data and formulae provided above.

Component	Value (m)
x_B	0.3
z_B	0.6
x_S	0.9067915590623501
z_S	0.2946344379171024
x_C	0.02610059280343250
z_C	-0.1023073115806087
x_D	0.03207142672761934
z_D	0.2676445084476887

Configuration: General Position

The figure to the right shows the bicycle in a more ***general configuration***. The figure illustrates ***five*** of the six angles used in the analysis. The rear frame B is oriented relative to the inertial frame R using a 3-1-2 orientation angle sequence. The steering column C and rear wheel are oriented relative to B by single angles, and the front wheel is oriented relative to C by a single angle.



The 3-1-2 orientation angles (ψ,ϕ,θ) orient the rear frame relative to R. They represent the ***yaw***, ***roll***, and ***pitch*** angles of B. The diagram shows ***positive*** yaw and roll angles. The pitch angle is not shown. The value of the pitch angle is small and is determined to ensure the front wheel remains in contact with the horizontal surface. The diagram also shows a ***positive steering angle*** δ. So, for the angles shown, the bicycle is ***rolling*** and ***turning*** to the ***right***. As noted earlier, the angles of the wheels relative to the frame are both negative for forward motion.

Degrees of Freedom:

One approach to determining the ***number of degrees of freedom*** of the bicycle is to first assume the frame B has six degrees of freedom. The front and rear wheels and the steering column all add one additional degree of freedom associated with the angles θ_F, θ_R, and δ. This makes a total of ***nine possible degrees of freedom***. However, there are ***six constraint equations*** associated with the ***no-slip conditions*** at points *P* and *Q*. Hence, the bicycle has only ***three degrees of freedom***. This, of course, assumes ***no rider motion*** relative to the bicycle.

In the analysis that follows (and consistent with other published literature), the angles ϕ, θ_R, and δ will be taken as an ***independent set*** of ***generalized coordinates*** representing the ***three*** degrees of freedom. It is also ***implicitly assumed*** that the ***velocity*** of *P* is ***zero***. This allows the equations of motion to be developed in terms of the six angles $(\psi,\phi,\theta,\theta_R,\delta,\theta_F)$ and their time derivatives. Using this approach, only ***three constraint equations*** are required associated with the ***no-slip*** condition at *Q*.

Kane's Equations of Motion:

Defining the vector of ***independent generalized speeds*** as $\{u\} \triangleq \{u_I\} = \begin{bmatrix} \dot{\phi} & \dot{\theta}_R & \dot{\delta} \end{bmatrix}^T$, Kane's equations of motion can be written as follows.

$$\boxed{\sum_{i=1}^{4}\left(m_i\, {}^R\underset{\sim}{a}_{G_i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k} \right) + \sum_{i=1}^{4}\left[\left(\underset{\approx}{I}_{G_i} \cdot {}^R\underset{\sim}{\alpha}_{B_i} \right) + \left({}^R\underset{\sim}{\omega}_{B_i} \times \underset{\sim}{H}_{G_i} \right) \right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k} = F_{u_k}} \qquad (k = 1,2,3)$$

Bodies one through four are the bodies *A* through *D* as previously defined. Note, in this form, these equations are equivalent to d'Alembert's Principle.

Details required to form the equations of motion are ***provided*** in the sections below. Specific results are provided for the transformation matrices, no-slip constraint equations and their derivatives, partial angular velocity and partial velocity matrices and their derivatives, angular accelerations, mass-center accelerations, and the generalized forces. All contributing terms are then combined to form a ***single matrix equation of motion***. When ***combined*** with the ***constraint equations***, these form a ***complete set*** of ***governing differential equations of motion*** for the bicycle. The ***generalized mass matrix*** and ***right-side vectors*** are clearly defined.

Transformation Matrices:

Frame *B* is oriented ***relative*** to the ***inertial frame*** *R* using a 3-1-2 rotation sequence (ψ,ϕ,θ) representing the yaw, roll, and pitch of *B*. The transformation matrix associated with this sequence is

$$\boxed{\left[R_{R2B}\right] = \begin{bmatrix} C_\psi C_\theta - S_\psi S_\phi S_\theta & S_\psi C_\theta + C_\psi S_\phi S_\theta & -C_\phi S_\theta \\ -S_\psi C_\phi & C_\psi C_\phi & S_\phi \\ C_\psi S_\theta + S_\psi S_\phi C_\theta & S_\psi S_\theta - C_\psi S_\phi C_\theta & C_\phi C_\theta \end{bmatrix}}$$

Frame *C* is oriented ***relative*** to frame *B* using a 2-3 rotation sequence (λ,δ). The steering tilt angle λ is constant. The transformation matrix associated with this sequence is

$$\left[R_{B2C}\right]=\begin{bmatrix} C_\lambda C_\delta & S_\delta & -S_\lambda C_\delta \\ -C_\lambda S_\delta & C_\delta & S_\lambda S_\delta \\ S_\lambda & 0 & C_\lambda \end{bmatrix}$$

The transformation matrix associated with the complete sequence from R to C is

$$\left[R_{R2C}\right]=\left[R_{B2C}\right]\left[R_{R2B}\right]$$

Here, $\left[R_{R2B}\right]$ converts vector components in R to vector components in B, $\left[R_{B2C}\right]$ converts vector components in B to vectors components in C, and $\left[R_{R2C}\right]$ converts vector components in R to vector components in C. The transposes of these matrices perform the opposite tasks.

Front Wheel Rolling Constraint:

Assuming ***no slippage*** at Q the ***contact point*** of the front wheel with the surface, the velocity of Q is set to ***zero***. Starting from P the contact point on the rear wheel, the ***velocity constraint*** can be written as

$$\begin{aligned} \underset{\sim}{0} = {}^R\underset{\sim}{v}_Q &= {}^R\underset{\sim}{v}_{G_A/P} + {}^R\underset{\sim}{v}_{S/G_A} + {}^R\underset{\sim}{v}_{G_D/S} + {}^R\underset{\sim}{v}_{Q/G_D} \\ &= \left({}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/P}\right)+\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{S/G_A}\right)+\left({}^R\underset{\sim}{\omega}_C \times \underset{\sim}{r}_{G_D/S}\right)+\left({}^R\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{Q/G_D}\right) \end{aligned}$$

Angular Velocities

Using a 3-1-2 orientation angle sequence, the ***angular velocity*** of the ***rear frame*** B can be written as

$${}^R\underset{\sim}{\omega}_B = \left(\dot{\phi}C_\theta - \dot{\psi}C_\phi S_\theta\right)\underset{\sim}{b}_1 + \left(\dot{\theta}+\dot{\psi}S_\phi\right)\underset{\sim}{b}_2 + \left(\dot{\phi}S_\theta + \dot{\psi}C_\phi C_\theta\right)\underset{\sim}{b}_3 \triangleq \omega_{B1}\underset{\sim}{b}_1 + \omega_{B2}\underset{\sim}{b}_2 + \omega_{B3}\underset{\sim}{b}_3$$

Or, in matrix form,

$$\begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix}_B = \begin{bmatrix} -C_\phi S_\theta & C_\theta & 0 \\ S_\phi & 0 & 1 \\ C_\phi C_\theta & S_\theta & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\phi} \\ \dot{\theta} \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

Note that a ***subscript*** B is included on the vector of components to indicate these are ***B-frame components***. Note also the terms on the right side of the equation have been separated into two types of terms, those involving only the ***derivatives*** of the ***independent angles*** and those involving only the ***derivatives*** of the ***dependent angles***.

The ***angular velocity*** of the ***rear wheel*** A can be written using the ***summation rule*** for angular velocities.

$${}^R\underset{\sim}{\omega}_A = {}^R\underset{\sim}{\omega}_B + {}^B\underset{\sim}{\omega}_A = \omega_{B1}\underset{\sim}{b}_1 + \left(\omega_{B2}+\dot{\theta}_R\right)\underset{\sim}{b}_2 + \omega_{B3}\underset{\sim}{b}_3 \triangleq \omega_{A1}\underset{\sim}{b}_1 + \omega_{A2}\underset{\sim}{b}_2 + \omega_{A3}\underset{\sim}{b}_3$$

Or, in matrix form,

$$\begin{Bmatrix} \omega_{A1} \\ \omega_{A2} \\ \omega_{A3} \end{Bmatrix}_B = \begin{bmatrix} -C_\phi S_\theta & C_\theta & 0 \\ S_\phi & 0 & 1 \\ C_\phi C_\theta & S_\theta & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\phi} \\ \dot{\theta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\theta}_R \\ 0 \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 1 & 0 \\ S_\theta & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

Again, using the summation rule, the ***angular velocity*** of the ***front frame*** C can be written as

$$
{}^R\underset{\sim}{\omega}_C = {}^R\underset{\sim}{\omega}_B + {}^B\underset{\sim}{\omega}_C = \left[\omega_{B1}\underset{\sim}{b}_1 + \omega_{B2}\underset{\sim}{b}_2 + \omega_{B3}\underset{\sim}{b}_3\right] + \dot{\delta}\,\underset{\sim}{c}_3 = \begin{bmatrix}\omega_{B1} & \omega_{B2} & \omega_{B3}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{b}_1\\ \underset{\sim}{b}_2\\ \underset{\sim}{b}_3\end{Bmatrix} + \dot{\delta}\,\underset{\sim}{c}_3
$$

$$
= \begin{bmatrix}\omega_{B1} & \omega_{B2} & \omega_{B3}\end{bmatrix}\left[R_{C2B}\right]\begin{Bmatrix}\underset{\sim}{c}_1\\ \underset{\sim}{c}_2\\ \underset{\sim}{c}_3\end{Bmatrix} + \begin{bmatrix}0 & 0 & \dot{\delta}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{c}_1\\ \underset{\sim}{c}_2\\ \underset{\sim}{c}_3\end{Bmatrix} \triangleq \omega_{C1}\underset{\sim}{c}_1 + \omega_{C2}\underset{\sim}{c}_2 + \omega_{C3}\underset{\sim}{c}_3
$$

Or, in matrix form

$$
\begin{Bmatrix}\omega_{C1}\\ \omega_{C2}\\ \omega_{C3}\end{Bmatrix}_C \triangleq \left[R_{C2B}\right]^T\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix} + \begin{Bmatrix}0\\ 0\\ \dot{\delta}\end{Bmatrix} = \left[R_{B2C}\right]\left\{\begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\theta}_F\end{Bmatrix}\right\} + \begin{Bmatrix}0\\ 0\\ \dot{\delta}\end{Bmatrix}
$$

$$
\boxed{\begin{Bmatrix}\omega_{C1}\\ \omega_{C2}\\ \omega_{C3}\end{Bmatrix}_C = \left[\left[R_{B2C}\right]\begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \left[R_{B2C}\right]\begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\theta}_F\end{Bmatrix}}
$$

Finally, the ***angular velocity*** of the ***front wheel*** can be written as

$$
\boxed{{}^R\underset{\sim}{\omega}_D = {}^R\underset{\sim}{\omega}_C + {}^C\underset{\sim}{\omega}_D = \omega_{C1}\underset{\sim}{c}_1 + \left(\omega_{C2} + \dot{\theta}_F\right)\underset{\sim}{c}_2 + \omega_{C3}\underset{\sim}{c}_3 \triangleq \omega_{D1}\underset{\sim}{c}_1 + \omega_{D2}\underset{\sim}{c}_2 + \omega_{D3}\underset{\sim}{c}_3}
$$

Or, in matrix form,

$$
\begin{Bmatrix}\omega_{D1}\\ \omega_{D2}\\ \omega_{D3}\end{Bmatrix}_C \triangleq \left[R_{C2B}\right]^T\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix} + \begin{Bmatrix}0\\ \dot{\theta}_F\\ \dot{\delta}\end{Bmatrix} = \left[R_{B2C}\right]\left\{\begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\theta}_F\end{Bmatrix}\right\} + \begin{Bmatrix}0\\ \dot{\theta}_F\\ \dot{\delta}\end{Bmatrix}
$$

$$
\boxed{\begin{Bmatrix}\omega_{D1}\\ \omega_{D2}\\ \omega_{D3}\end{Bmatrix}_C = \left[\left[R_{B2C}\right]\begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \left[\left[R_{B2C}\right]\begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\right]\begin{Bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\theta}_F\end{Bmatrix}}
$$

<u>Relative Velocities</u>

The velocity of G_A relative to P can be written as

$$
\boxed{{}^R\underset{\sim}{v}_{G_A/P} = {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{G_A/P} = -\underset{\sim}{r}_{G_A/P} \times {}^R\underset{\sim}{\omega}_A \triangleq v_{A1}\underset{\sim}{b}_1 + v_{A2}\underset{\sim}{b}_2 + v_{A3}\underset{\sim}{b}_3}
$$

Using a ***skew symmetric matrix*** to perform the ***cross product***, the result can be written in matrix form as

$$\begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_B = -r_{RW} \begin{bmatrix} 0 & C_\theta & 0 \\ -C_\theta & 0 & -S_\theta \\ 0 & S_\theta & 0 \end{bmatrix} \begin{Bmatrix} \omega_{A1} \\ \omega_{A2} \\ \omega_{A3} \end{Bmatrix}_B$$

$$= r_{RW} \begin{bmatrix} 0 & -C_\theta & 0 \\ C_\theta & 0 & S_\theta \\ 0 & -S_\theta & 0 \end{bmatrix} \left\{ \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 1 & 0 \\ S_\theta & 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix} \right\}$$

$$\boxed{\begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_B = r_{RW} \begin{bmatrix} 0 & -C_\theta & 0 \\ 1 & 0 & 0 \\ 0 & -S_\theta & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + r_{RW} \begin{bmatrix} -C_\theta S_\phi & -C_\theta & 0 \\ 0 & 0 & 0 \\ -S_\theta S_\phi & -S_\theta & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}}$$

Note that $\underset{\sim}{r}_{G_A/P} = -r_{RW}\,\underset{\sim}{b}_3$ only when the pitch angle θ is zero. For non-zero θ, the vector must be broken into components along the $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_3$ directions. That is, $\underset{\sim}{r}_{G_A/P} = r_{RW}\left(-S_\theta\,\underset{\sim}{b}_1 + C_\theta\,\underset{\sim}{b}_3\right)$.

The velocity of S with respect to G_A can be written as

$$\boxed{{}^R\underset{\sim}{v}_{S/G_A} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{S/G_A} = -\underset{\sim}{r}_{S/G_A} \times {}^R\underset{\sim}{\omega}_B \triangleq v_{AS1}\,\underset{\sim}{b}_1 + v_{AS2}\,\underset{\sim}{b}_2 + v_{AS3}\,\underset{\sim}{b}_3}$$

Using a ***skew symmetric matrix*** to perform the cross product, the result can be written as

$$\begin{Bmatrix} v_{AS1} \\ v_{AS2} \\ v_{AS3} \end{Bmatrix}_B = -\begin{bmatrix} 0 & z_S & 0 \\ -z_S & 0 & -x_S \\ 0 & x_S & 0 \end{bmatrix} \begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix}_B = \begin{bmatrix} 0 & -z_S & 0 \\ z_S & 0 & x_S \\ 0 & -x_S & 0 \end{bmatrix} \left\{ \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix} \right\}$$

$$\boxed{\begin{Bmatrix} v_{AS1} \\ v_{AS2} \\ v_{AS3} \end{Bmatrix}_B = \begin{bmatrix} 0 & 0 & 0 \\ \left(x_S S_\theta + z_S C_\theta\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -z_S S_\phi & -z_S & 0 \\ \left(x_S C_\theta - z_S S_\theta\right) C_\phi & 0 & 0 \\ -x_S S_\phi & -x_S & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}}$$

The velocity of G_D relative to S can be written as

$$\boxed{{}^R\underset{\sim}{v}_{G_D/S} = {}^R\underset{\sim}{\omega}_C \times \underset{\sim}{r}_{G_D/S} = -\underset{\sim}{r}_{G_D/S} \times {}^R\underset{\sim}{\omega}_C \triangleq v_{SD1}\,\underset{\sim}{c}_1 + v_{SD2}\,\underset{\sim}{c}_2 + v_{SD3}\,\underset{\sim}{c}_3}$$

Again, using a skew symmetric matrix to perform the cross product, the result can be written as

$$\begin{Bmatrix} v_{SD1} \\ v_{SD2} \\ v_{SD3} \end{Bmatrix}_C = -\begin{bmatrix} 0 & -z_D & 0 \\ z_D & 0 & -x_D \\ 0 & x_D & 0 \end{bmatrix} \begin{Bmatrix} \omega_{C1} \\ \omega_{C2} \\ \omega_{C3} \end{Bmatrix}_C$$

$$= \begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} \left[\left[R_{B2C} \right] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} \left[R_{B2C} \right] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

To calculate the velocity of Q with respect to G_D, the ***position vector*** of the ***contact point*** Q relative to the ***mass center*** G_D must be found. For an ***arbitrary orientation*** of the ***front wheel***, it can be shown that

$$\underset{\sim}{r}_{Q/G_D} = r_{FW} \frac{\left[\underset{\sim}{c}_2 \times \left(\underset{\sim}{N}_3 \times \underset{\sim}{c}_2 \right) \right]}{\left\| \underset{\sim}{c}_2 \times \left(\underset{\sim}{N}_3 \times \underset{\sim}{c}_2 \right) \right\|} = r_{FW} \frac{\left[\left(\underset{\sim}{c}_2 \cdot \underset{\sim}{c}_2 \right) \underset{\sim}{N}_3 - \left(\underset{\sim}{N}_3 \cdot \underset{\sim}{c}_2 \right) \underset{\sim}{c}_2 \right]}{\left\| \underset{\sim}{c}_2 \times \left(\underset{\sim}{N}_3 \times \underset{\sim}{c}_2 \right) \right\|}$$

Now, if $\underset{\sim}{N}_3 = n_{31} \underset{\sim}{c}_1 + n_{32} \underset{\sim}{c}_2 + n_{33} \underset{\sim}{c}_3$, then

$$\left(\underset{\sim}{c}_2 \cdot \underset{\sim}{c}_2 \right) \underset{\sim}{N}_3 - \left(\underset{\sim}{N}_3 \cdot \underset{\sim}{c}_2 \right) \underset{\sim}{c}_2 = n_{31} \underset{\sim}{c}_1 + n_{33} \underset{\sim}{c}_3 \quad \text{and} \quad \left\| \underset{\sim}{c}_2 \times \left(\underset{\sim}{N}_3 \times \underset{\sim}{c}_2 \right) \right\| = \sqrt{n_{31}^2 + n_{33}^2}$$

Substituting into the expression for $\underset{\sim}{r}_{Q/G_D}$ gives

$$\underset{\sim}{r}_{Q/G_D} = \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}} \left(n_{31} \underset{\sim}{c}_1 + n_{33} \underset{\sim}{c}_3 \right)$$

The ***velocity*** of the ***contact point*** Q ***relative*** to G_D can then be written as

$${}^R\underset{\sim}{v}_{Q/G_D} = {}^R\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{Q/G_D} = -\underset{\sim}{r}_{Q/G_D} \times {}^R\underset{\sim}{\omega}_D \triangleq v_{DQ1} \underset{\sim}{c}_1 + v_{DQ2} \underset{\sim}{c}_2 + v_{DQ3} \underset{\sim}{c}_3$$

The result can be written as

$$\begin{Bmatrix} v_{DQ1} \\ v_{DQ2} \\ v_{DQ3} \end{Bmatrix}_C = -\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}} \begin{bmatrix} 0 & -n_{33} & 0 \\ n_{33} & 0 & -n_{31} \\ 0 & n_{31} & 0 \end{bmatrix} \begin{Bmatrix} \omega_{D1} \\ \omega_{D2} \\ \omega_{D3} \end{Bmatrix}_C$$

$$= \frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}} \begin{bmatrix} 0 & n_{33} & 0 \\ -n_{33} & 0 & n_{31} \\ 0 & -n_{31} & 0 \end{bmatrix} \left[\left[R_{B2C}\right] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} +$$

$$\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}} \begin{bmatrix} 0 & n_{33} & 0 \\ -n_{33} & 0 & n_{31} \\ 0 & -n_{31} & 0 \end{bmatrix} \left[\left[R_{B2C}\right] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right] \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

Or,

$$\begin{Bmatrix} v_{DQ1} \\ v_{DQ2} \\ v_{DQ3} \end{Bmatrix}_C = \frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}} \left[\begin{bmatrix} 0 & n_{33} & 0 \\ -n_{33} & 0 & n_{31} \\ 0 & -n_{31} & 0 \end{bmatrix} \left[R_{B2C}\right] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & n_{31} \\ 0 & 0 & 0 \end{bmatrix} \right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} +$$

$$\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}} \left[\begin{bmatrix} 0 & n_{33} & 0 \\ -n_{33} & 0 & n_{31} \\ 0 & -n_{31} & 0 \end{bmatrix} \left[R_{B2C}\right] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & n_{33} \\ 0 & 0 & 0 \\ 0 & 0 & -n_{31} \end{bmatrix} \right] \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

Substituting the above results into the ***front wheel rolling constraint*** gives ***three scalar constraint equations*** that can be written as

$$\left[C_1\right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \left[C_2\right] \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The constraint matrices $\left[C_1\right]$ and $\left[C_2\right]$ are defined as follows.

$$\left[C_1\right] \triangleq \left[R_{B2C}\right] \begin{bmatrix} 0 & -r_{RW}C_\theta & 0 \\ r_{RW}+x_S S_\theta + z_S C_\theta & 0 & 0 \\ 0 & -r_{RW}S_\theta & 0 \end{bmatrix} + \begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} \left[R_{B2C}\right] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}} \left[\begin{bmatrix} 0 & n_{33} & 0 \\ -n_{33} & 0 & n_{31} \\ 0 & -n_{31} & 0 \end{bmatrix} \left[R_{B2C}\right] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & n_{31} \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$[C_2] \triangleq [R_{B2C}] \begin{bmatrix} -(r_{RW}C_\theta + z_S)S_\phi & -(r_{RW}C_\theta + z_S) & 0 \\ (x_S C_\theta - z_S S_\theta)C_\phi & 0 & 0 \\ -(r_{RW}S_\theta + x_S)S_\phi & -(r_{RW}S_\theta + x_S) & 0 \end{bmatrix} + \begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} [R_{B2C}] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} +$$
$$\frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}} \left[\begin{bmatrix} 0 & n_{33} & 0 \\ -n_{33} & 0 & n_{31} \\ 0 & -n_{31} & 0 \end{bmatrix} [R_{B2C}] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & n_{33} \\ 0 & 0 & 0 \\ 0 & 0 & -n_{31} \end{bmatrix} \right]$$

Given a set of ***independent generalized speeds***, these equations can be used to ***calculate*** the ***dependent speeds***. Specifically,

$$\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix} = -[C_2]^{-1}[C_1] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} \triangleq [J] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

To make the equation more compact, define $\{u_D\} = \begin{bmatrix} \dot{\psi} & \dot{\theta} & \dot{\theta}_F \end{bmatrix}^T$ and $\{u_I\} = \begin{bmatrix} \dot{\phi} & \dot{\theta}_R & \dot{\delta} \end{bmatrix}^T$ to get

$$\{u_D\} = [J]\{u_I\}$$

Note the elements of the matrix $[J]$ represent the ***partial derivatives*** of the ***dependent generalized speeds*** with respect to the ***independent ones***. That is,

$$\frac{\partial u_{Di}}{\partial u_{Ij}} = J_{ij}$$

Note: The forms of the matrices $[C_1]$ and $[C_2]$ shown above can be ***further simplified*** by expansion of the matrix products while ***taking advantage*** of the ***numerous zero elements***.

Time Derivatives of the Constraint Matrices:

The derivatives of the dependent generalized speeds can be related to the derivatives of the independent generalized speeds by ***differentiating*** the ***constraint equation***.

$$\{\dot{u}_D\} = [J]\{\dot{u}_I\} + [\dot{J}]\{u_I\}$$

The derivative of the constraint matrix $[J]$ can be determined by differentiating the unsolved constraint equation. That is,

$$[C_1]\{u_I\} + [C_2]\{u_D\} = \{0\} \qquad \Rightarrow \qquad [C_1]\{\dot{u}_I\} + [C_2]\{\dot{u}_D\} + [\dot{C}_1]\{u_I\} + [\dot{C}_2]\{u_D\} = \{0\}$$

Solving for the derivatives of the dependent speeds gives

$$\boxed{\begin{aligned}\{\dot{u}_D\} &= \left[-[C_2]^{-1}[C_1]\right]\{\dot{u}_I\} + \left[-[C_2]^{-1}\left[\left[\dot{C}_1\right]-\left[\dot{C}_2\right][J]\right]\right]\{u_I\} \\ &\triangleq [J]\{\dot{u}_I\} + \left[\dot{J}\right]\{u_I\}\end{aligned}}$$

Examining the above result gives

$$\boxed{\left[\dot{J}\right] = -[C_2]^{-1}\left[\left[\dot{C}_1\right]-\left[\dot{C}_2\right][J]\right]}$$

<u>Time Derivatives of the C-frame Components of $\underset{\sim}{N}_3$</u>

The C-frame components of the unit vector $\underset{\sim}{N}_3$ can be found using the ***transformation matrices***.

$$\boxed{\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix} = \left[R_{R2C}\right]^T \begin{Bmatrix}\underset{\sim}{c}_1\\ \underset{\sim}{c}_2\\ \underset{\sim}{c}_3\end{Bmatrix} = \left(\left[R_{B2C}\right]\left[R_{R2B}\right]\right)^T \begin{Bmatrix}\underset{\sim}{c}_1\\ \underset{\sim}{c}_2\\ \underset{\sim}{c}_3\end{Bmatrix} = \left[R_{R2B}\right]^T\left[R_{B2C}\right]^T \begin{Bmatrix}\underset{\sim}{c}_1\\ \underset{\sim}{c}_2\\ \underset{\sim}{c}_3\end{Bmatrix}}$$

$$\Rightarrow \boxed{\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix} = \begin{bmatrix} C_\psi C_\theta - S_\psi S_\phi S_\theta & -S_\psi C_\phi & C_\psi S_\theta + S_\psi S_\phi C_\theta \\ S_\psi C_\theta + C_\psi S_\phi S_\theta & C_\psi C_\phi & S_\psi S_\theta - C_\psi S_\phi C_\theta \\ -C_\phi S_\theta & S_\phi & C_\phi C_\theta \end{bmatrix} \begin{bmatrix} C_\lambda C_\delta & -C_\lambda S_\delta & S_\lambda \\ S_\delta & C_\delta & 0 \\ -S_\lambda C_\delta & S_\lambda S_\delta & C_\lambda \end{bmatrix} \begin{Bmatrix}\underset{\sim}{c}_1\\ \underset{\sim}{c}_2\\ \underset{\sim}{c}_3\end{Bmatrix}}$$

Using this result, the C-frame components of unit vector $\underset{\sim}{N}_3$ can be written as

$$\boxed{\begin{Bmatrix}\underset{\sim}{N}_3\cdot\underset{\sim}{c}_1\\ \underset{\sim}{N}_3\cdot\underset{\sim}{c}_2\\ \underset{\sim}{N}_3\cdot\underset{\sim}{c}_3\end{Bmatrix} = \begin{Bmatrix}n_{31}\\ n_{32}\\ n_{33}\end{Bmatrix} = \begin{Bmatrix} -C_\phi S_\theta C_\lambda C_\delta + S_\phi S_\delta - C_\phi C_\theta S_\lambda C_\delta \\ C_\phi S_\theta C_\lambda S_\delta + S_\phi C_\delta + C_\phi C_\theta S_\lambda S_\delta \\ -C_\phi S_\theta S_\lambda + C_\phi C_\theta C_\lambda \end{Bmatrix}}$$

The ***derivatives*** of the ***first*** and ***third*** components can be written as follows

$$\begin{aligned}\dot{n}_{31} &= C_\lambda\left[\left(S_\phi C_\delta S_\theta\right)\dot{\phi} + \left(C_\phi S_\delta S_\theta\right)\dot{\delta} - \left(C_\phi C_\delta C_\theta\right)\dot{\theta}\right] + \left(C_\phi S_\delta\right)\dot{\phi} + \left(S_\phi C_\delta\right)\dot{\delta} \\ &\quad + S_\lambda\left[\left(S_\phi C_\delta C_\theta\right)\dot{\phi} + \left(C_\phi S_\delta C_\theta\right)\dot{\delta} + \left(C_\phi C_\delta S_\theta\right)\dot{\theta}\right] \\ &= \left(S_\phi C_\delta\left(C_\lambda S_\theta + S_\lambda C_\theta\right) + C_\phi S_\delta\right)\dot{\phi} + \left(C_\phi S_\delta\left(C_\lambda S_\theta + S_\lambda C_\theta\right) + S_\phi C_\delta\right)\dot{\delta} - C_\phi C_\delta\left(C_\lambda C_\theta - S_\lambda S_\theta\right)\dot{\theta}\end{aligned}$$

$$\boxed{\dot{n}_{31} = \left(S_\phi C_\delta S_{\lambda+\theta} + C_\phi S_\delta\right)\dot{\phi} + \left(C_\phi S_\delta S_{\lambda+\theta} + S_\phi C_\delta\right)\dot{\delta} - C_\phi C_\delta C_{\lambda+\theta}\dot{\theta}}$$

$$\begin{aligned}\dot{n}_{33} &= -S_\lambda\left(-\left(S_\phi S_\theta\right)\dot{\phi} + \left(C_\phi C_\theta\right)\dot{\theta}\right) + C_\lambda\left(-\left(S_\phi C_\theta\right)\dot{\phi} - \left(C_\phi S_\theta\right)\dot{\theta}\right) \\ &= -S_\phi\left(C_\lambda C_\theta - S_\lambda S_\theta\right)\dot{\phi} - C_\phi\left(S_\lambda C_\theta + C_\lambda S_\theta\right)\dot{\theta}\end{aligned}$$

$$\boxed{\dot{n}_{33} = -S_\phi C_{\lambda+\theta}\dot{\phi} - C_\phi S_{\lambda+\theta}\dot{\theta}}$$

Note that the symbols $C_{\lambda+\theta}$ and $S_{\lambda+\theta}$ are used to represent the cosine and sine of the ***double angle*** $\lambda+\theta$. These expressions are useful when calculating the time derivatives of the constraint matrices.

Time Derivatives of Constraint Matrices $\left[C_1\right]$ and $\left[C_2\right]$

To calculate the ***derivatives*** of matrices $\left[C_1\right]$ and $\left[C_2\right]$, it is first helpful to use ***simplified expressions***. Expanding the expressions given above, it can be shown that $\left[C_1\right]$ and $\left[C_2\right]$ can be written as

$$\left[C_1\right]=\left[C_{1A}\right]+\left[C_{1B}\right]+\left[C_{1C}\right]$$

$$=\begin{bmatrix} S_\delta\left(r_{RW}+\left(x_S S_\theta+z_S C_\theta\right)\right) & -r_{RW}C_\delta C_{\lambda+\theta} & 0 \\ C_\delta\left(r_{RW}+\left(x_S S_\theta+z_S C_\theta\right)\right) & r_{RW}S_\delta C_{\lambda+\theta} & 0 \\ 0 & -r_{RW}S_{\lambda+\theta} & 0 \end{bmatrix}+\begin{bmatrix} -z_D S_\delta C_{\lambda+\theta} & 0 & 0 \\ x_D S_{\lambda+\theta}-z_D C_\delta C_{\lambda+\theta} & 0 & x_D \\ x_D S_\delta C_{\lambda+\theta} & 0 & 0 \end{bmatrix}$$

$$+\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}}\begin{bmatrix} -n_{33}S_\delta C_{\lambda+\theta} & 0 & 0 \\ -n_{33}C_\delta C_{\lambda+\theta}+n_{31}S_{\lambda+\theta} & 0 & n_{31} \\ n_{31}S_\delta C_{\lambda+\theta} & 0 & 0 \end{bmatrix}$$

$$\left[C_2\right]=\begin{bmatrix} \left(C_{2A}\right)_{11} & \left(C_{2A}\right)_{12} & 0 \\ \left(C_{2A}\right)_{21} & \left(C_{2A}\right)_{22} & 0 \\ \left(C_{2A}\right)_{31} & \left(C_{2A}\right)_{32} & 0 \end{bmatrix}+\begin{bmatrix} \left(C_{2B}\right)_{11} & \left(C_{2B}\right)_{12} & 0 \\ \left(C_{2B}\right)_{21} & \left(C_{2B}\right)_{22} & 0 \\ \left(C_{2B}\right)_{31} & \left(C_{2B}\right)_{32} & 0 \end{bmatrix}+\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}}\begin{bmatrix} \left(\tilde{C}_{2C}\right)_{11} & \left(\tilde{C}_{2C}\right)_{12} & n_{33} \\ \left(\tilde{C}_{2C}\right)_{21} & \left(\tilde{C}_{2C}\right)_{22} & 0 \\ \left(\tilde{C}_{2C}\right)_{31} & \left(\tilde{C}_{2C}\right)_{32} & -n_{31} \end{bmatrix}$$

The ***non-zero*** entries of the $\left[C_2\right]$ submatrices are as follows.

$$\left(C_{2A}\right)_{11}=-r_{RW}C_\delta S_\phi C_{\lambda+\theta}+\left(x_S S_\lambda-z_S C_\lambda\right)C_\delta S_\phi+\left(x_S C_\theta-z_S S_\theta\right)S_\delta C_\phi$$

$$\left(C_{2A}\right)_{21}=r_{RW}S_\delta S_\phi C_{\lambda+\theta}+\left(z_S C_\lambda-x_S S_\lambda\right)S_\delta S_\phi+\left(x_S C_\theta-z_S S_\theta\right)C_\delta C_\phi$$

$$\left(C_{2A}\right)_{31}=-r_{RW}S_\phi S_{\lambda+\theta}-\left(x_S C_\lambda+z_S S_\lambda\right)S_\phi \qquad \left(C_{2A}\right)_{12}=-r_{RW}C_\delta C_{\lambda+\theta}+\left(x_S S_\lambda-z_S C_\lambda\right)C_\delta$$

$$\left(C_{2A}\right)_{22}=r_{RW}S_\delta C_{\lambda+\theta}+\left(z_S C_\lambda-x_S S_\lambda\right)S_\delta \qquad \left(C_{2A}\right)_{32}=-r_{RW}S_{\lambda+\theta}-\left(x_S C_\lambda+z_S S_\lambda\right)$$

$$\left(C_{2B}\right)_{11}=z_D S_{\lambda+\theta}S_\delta C_\phi+z_D C_\delta S_\phi \qquad \left(C_{2B}\right)_{21}=z_D S_{\lambda+\theta}C_\phi C_\delta-z_D S_\delta S_\phi+x_D C_{\lambda+\theta}C_\phi$$

$$\left(C_{2B}\right)_{31}=-x_D S_{\lambda+\theta}S_\delta C_\phi-x_D C_\delta S_\phi \qquad \left(C_{2B}\right)_{12}=z_D C_\delta \qquad \left(C_{2B}\right)_{22}=-z_D S_\delta \qquad \left(C_{2B}\right)_{32}=-x_D C_\delta$$

$$\left(\tilde{C}_{2C}\right)_{11}=n_{33}S_{\lambda+\theta}S_\delta C_\phi+n_{33}C_\delta S_\phi \qquad \left(\tilde{C}_{2C}\right)_{21}=n_{31}C_{\lambda+\theta}C_\phi+n_{33}S_{\lambda+\theta}C_\delta C_\phi-n_{33}S_\delta S_\phi$$

$$\left(\tilde{C}_{2C}\right)_{31}=-n_{31}S_{\lambda+\theta}S_\delta C_\phi-n_{31}C_\delta S_\phi \qquad \left(\tilde{C}_{2C}\right)_{12}=n_{33}C_\delta$$

$$\left(\tilde{C}_{2C}\right)_{22}=-n_{33}S_\delta \qquad \left(\tilde{C}_{2C}\right)_{32}=-n_{31}C_\delta$$

Differentiating the above expressions, the ***non-zero elements*** of the matrices $\left[\dot{C}_{1A}\right]$ and $\left[\dot{C}_{1B}\right]$ can be written as follows.

$$\left(\dot{C}_{1A}\right)_{11} = \left(r_{RW} + \left(x_S S_\theta + z_S C_\theta\right)\right) C_\delta \dot{\delta} + S_\delta \left(x_S C_\theta - z_S S_\theta\right) \dot{\theta}$$

$$\left(\dot{C}_{1A}\right)_{21} = -\left(r_{RW} + \left(x_S S_\theta + z_S C_\theta\right)\right) S_\delta \dot{\delta} + C_\delta \left(x_S C_\theta - z_S S_\theta\right) \dot{\theta}$$

$$\left(\dot{C}_{1A}\right)_{12} = \left(r_{RW} S_\delta C_{\lambda+\theta}\right)\dot{\delta} + \left(r_{RW} C_\delta S_{\lambda+\theta}\right)\dot{\theta} \qquad \left(\dot{C}_{1A}\right)_{22} = \left(r_{RW} C_\delta C_{\lambda+\theta}\right)\dot{\delta} - \left(r_{RW} S_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{1A}\right)_{32} = \left(-r_{RW} C_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{1B}\right)_{11} = \left(-z_D C_\delta C_{\lambda+\theta}\right)\dot{\delta} + \left(z_D S_\delta S_{\lambda+\theta}\right)\dot{\theta} \qquad \left(\dot{C}_{1B}\right)_{21} = \left(z_D S_\delta C_{\lambda+\theta}\right)\dot{\delta} + \left(x_D C_{\lambda+\theta} + z_D C_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{1B}\right)_{31} = \left(x_D C_\delta C_{\lambda+\theta}\right)\dot{\delta} - \left(x_D S_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

Using the product rule, the matrix $\left[\dot{C}_{1C}\right]$ can be written as follows.

$$\left[\dot{C}_{1C}\right] = \frac{d}{dt}\left(\frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\right)\begin{bmatrix} -n_{33} S_\delta C_{\lambda+\theta} & 0 & 0 \\ -n_{33} C_\delta C_{\lambda+\theta} + n_{31} S_{\lambda+\theta} & 0 & n_{31} \\ n_{31} S_\delta C_{\lambda+\theta} & 0 & 0 \end{bmatrix} + \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\begin{bmatrix} \frac{d}{dt}\left(-n_{33} S_\delta C_{\lambda+\theta}\right) & 0 & 0 \\ \frac{d}{dt}\left(-n_{33} C_\delta C_{\lambda+\theta} + n_{31} S_{\lambda+\theta}\right) & 0 & \dot{n}_{31} \\ \frac{d}{dt}\left(n_{31} S_\delta C_{\lambda+\theta}\right) & 0 & 0 \end{bmatrix}$$

Here,

$$\frac{d}{dt}\left(\frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\right) = r_{FW}\frac{d}{dt}\left(\left(n_{31}^2 + n_{33}^2\right)^{-\frac{1}{2}}\right) = -\tfrac{1}{2} r_{FW}\left(n_{31}^2 + n_{33}^2\right)^{-\frac{3}{2}}\left(2n_{31}\dot{n}_{31} + 2n_{33}\dot{n}_{33}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\right) = \frac{-r_{FW}\left(n_{31}\dot{n}_{31} + n_{33}\dot{n}_{33}\right)}{\left(n_{31}^2 + n_{33}^2\right)^{\frac{3}{2}}}$$

$$\frac{d}{dt}\left(-n_{33} S_\delta C_{\lambda+\theta}\right) = -\dot{n}_{33} S_\delta C_{\lambda+\theta} - \left(n_{33} C_\delta C_{\lambda+\theta}\right)\dot{\delta} + \left(n_{33} S_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

$$\frac{d}{dt}\left(-n_{33} C_\delta C_{\lambda+\theta} + n_{31} S_{\lambda+\theta}\right) = -\dot{n}_{33} C_\delta C_{\lambda+\theta} + \left(n_{33} S_\delta C_{\lambda+\theta}\right)\dot{\delta} + \left(n_{33} C_\delta S_{\lambda+\theta}\right)\dot{\theta} + \dot{n}_{31} S_{\lambda+\theta} + \left(n_{31} C_{\lambda+\theta}\right)\dot{\theta}$$

$$\frac{d}{dt}\left(n_{31} S_\delta C_{\lambda+\theta}\right) = \dot{n}_{31} S_\delta C_{\lambda+\theta} + \left(n_{31} C_\delta C_{\lambda+\theta}\right)\dot{\delta} - \left(n_{31} S_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

The time derivative of matrix $\left[C_2\right]$ can be calculated as follows.

$$\left[\dot{C}_2\right] = \left[\dot{C}_{2A}\right] + \left[\dot{C}_{2B}\right] + \left[\dot{C}_{2C}\right]$$

The non-zero elements of $\left[\dot{C}_{2A}\right]$ can be written as

$$\begin{aligned}\left(\dot{C}_{2A}\right)_{11} = &\left[\left(-r_{RW}C_\phi C_\delta C_{\lambda+\theta}\right)+\left(x_S S_\lambda - z_S C_\lambda\right)\left(C_\phi C_\delta\right)-\left(x_S C_\theta - z_S S_\theta\right)\left(S_\phi S_\delta\right)\right]\dot{\phi} \\ &+\left[\left(r_{RW}S_\phi S_\delta C_{\lambda+\theta}\right)-\left(x_S S_\lambda - z_S C_\lambda\right)\left(S_\phi S_\delta\right)+\left(x_S C_\theta - z_S S_\theta\right)\left(C_\phi C_\delta\right)\right]\dot{\delta} \\ &-\left[\left(x_S S_\theta + z_S C_\theta\right)\left(C_\phi S_\delta\right)+\left(r_{RW}S_\phi C_\delta S_{\lambda+\theta}\right)\right]\dot{\theta}\end{aligned}$$

$$\begin{aligned}\left(\dot{C}_{2A}\right)_{21} = &\left[\left(r_{RW}C_\phi S_\delta C_{\lambda+\theta}\right)-\left(x_S S_\lambda - z_S C_\lambda\right)\left(C_\phi S_\delta\right)-\left(x_S C_\theta - z_S S_\theta\right)\left(S_\phi C_\delta\right)\right]\dot{\phi} \\ &+\left[\left(r_{RW}S_\phi C_\delta C_{\lambda+\theta}\right)-\left(x_S S_\lambda - z_S C_\lambda\right)\left(S_\phi C_\delta\right)-\left(x_S C_\theta - z_S S_\theta\right)\left(C_\phi S_\delta\right)\right]\dot{\delta} \\ &-\left[\left(x_S S_\theta + z_S C_\theta\right)\left(C_\phi C_\delta\right)+\left(r_{RW}S_\phi S_\delta S_{\lambda+\theta}\right)\right]\dot{\theta}\end{aligned}$$

$$\left(\dot{C}_{2A}\right)_{31} = -\left[\left(r_{RW}S_{\lambda+\theta}\right)+\left(x_S C_\lambda + z_S S_\lambda\right)\right]\left(C_\phi\right)\dot{\phi}-\left(r_{RW}S_\phi C_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{2A}\right)_{12} = \left[\left(r_{RW}C_{\lambda+\theta}\right)-\left(x_S S_\lambda - z_S C_\lambda\right)\right]\left(S_\delta\right)\dot{\delta}+\left(r_{RW}C_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{2A}\right)_{22} = \left[\left(r_{RW}C_{\lambda+\theta}\right)-\left(x_S S_\lambda - z_S C_\lambda\right)\right]\left(C_\delta\right)\dot{\delta}-\left(r_{RW}S_\delta S_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{2A}\right)_{32} = -\left(r_{RW}C_{\lambda+\theta}\right)\dot{\theta}$$

The non-zero elements of $\left[\dot{C}_{2B}\right]$ can be written as

$$\left(\dot{C}_{2B}\right)_{11} = z_D\left[-\left(S_\phi S_\delta S_{\lambda+\theta}\right)+\left(C_\phi C_\delta\right)\right]\dot{\phi}+z_D\left[\left(C_\phi C_\delta S_{\lambda+\theta}\right)-\left(S_\phi S_\delta\right)\right]\dot{\delta}+z_D\left(C_\phi S_\delta C_{\lambda+\theta}\right)\dot{\theta}$$

$$\begin{aligned}\left(\dot{C}_{2B}\right)_{21} = &-\left[z_D\left(S_\phi C_\delta S_{\lambda+\theta}\right)+z_D\left(C_\phi S_\delta\right)+x_D\left(S_\phi C_{\lambda+\theta}\right)\right]\dot{\phi}-z_D\left[\left(C_\phi S_\delta S_{\lambda+\theta}\right)+\left(S_\phi C_\delta\right)\right]\dot{\delta} \\ &+\left[z_D\left(C_\delta C_{\lambda+\theta}\right)-x_D\left(S_{\lambda+\theta}\right)\right]\left(C_\phi\right)\dot{\theta}\end{aligned}$$

$$\left(\dot{C}_{2B}\right)_{31} = x_D\left[\left(S_\phi S_\delta S_{\lambda+\theta}\right)-\left(C_\phi C_\delta\right)\right]\dot{\phi}+x_D\left[\left(S_\phi S_\delta\right)-\left(C_\phi C_\delta S_{\lambda+\theta}\right)\right]\dot{\delta}-x_D\left(C_\phi S_\delta C_{\lambda+\theta}\right)\dot{\theta}$$

$$\left(\dot{C}_{2B}\right)_{12} = -\left(z_D S_\delta\right)\dot{\delta} \qquad \left(\dot{C}_{2B}\right)_{22} = -\left(z_D C_\delta\right)\dot{\delta} \qquad \left(\dot{C}_{2B}\right)_{32} = \left(x_D S_\delta\right)\dot{\delta}$$

Using the product rule, the matrix $\left[\dot{C}_{2C}\right]$ can be written as

$$\left[\dot{C}_{2C}\right] = \frac{d}{dt}\left(\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}}\right)\begin{bmatrix}\left(\tilde{C}_{2C}\right)_{11} & \left(\tilde{C}_{2C}\right)_{12} & n_{33} \\ \left(\tilde{C}_{2C}\right)_{21} & \left(\tilde{C}_{2C}\right)_{22} & 0 \\ \left(\tilde{C}_{2C}\right)_{31} & \left(\tilde{C}_{2C}\right)_{32} & -n_{31}\end{bmatrix}+\frac{r_{FW}}{\sqrt{n_{31}^2+n_{33}^2}}\begin{bmatrix}\left(\dot{\tilde{C}}_{2C}\right)_{11} & \left(\dot{\tilde{C}}_{2C}\right)_{12} & \dot{n}_{33} \\ \left(\dot{\tilde{C}}_{2C}\right)_{21} & \left(\dot{\tilde{C}}_{2C}\right)_{22} & 0 \\ \left(\dot{\tilde{C}}_{2C}\right)_{31} & \left(\dot{\tilde{C}}_{2C}\right)_{32} & -\dot{n}_{31}\end{bmatrix}$$

The first matrix is well defined given the previous results, and the derivatives of the entries in the second term are as follows.

$$\left(\dot{\tilde{C}}_{2C}\right)_{11} = n_{33}\left(C_\phi C_\delta - S_\phi S_\delta S_{\lambda+\theta}\right)\dot{\phi} + n_{33}\left(C_\phi C_\delta S_{\lambda+\theta} - S_\phi S_\delta\right)\dot{\delta} + n_{33}\left(C_\phi S_\delta C_{\lambda+\theta}\right)\dot{\theta} + \dot{n}_{33}\left(C_\phi S_\delta S_{\lambda+\theta} + S_\phi C_\delta\right)$$

$$\left(\dot{\tilde{C}}_{2C}\right)_{21} = -\left[n_{31}\left(S_\phi C_{\lambda+\theta}\right) + n_{33}\left(S_\phi C_\delta S_{\lambda+\theta}\right) + n_{33}\left(C_\phi S_\delta\right)\right]\dot{\phi} - n_{33}\left[C_\phi S_\delta S_{\lambda+\theta} + S_\phi C_\delta\right]\dot{\delta} + \left[n_{33}\left(C_\delta C_{\lambda+\theta}\right) - n_{31}\left(S_{\lambda+\theta}\right)\right]\left(C_\phi\right)\dot{\theta} + \dot{n}_{31} C_\phi C_{\lambda+\theta} + \dot{n}_{33}\left[C_\phi C_\delta S_{\lambda+\theta} - S_\phi S_\delta\right]$$

$$\left(\dot{\tilde{C}}_{2C}\right)_{31} = n_{31}\left(S_\phi S_\delta S_{\lambda+\theta} - C_\phi C_\delta\right)\dot{\phi} + n_{31}\left(S_\phi S_\delta - C_\phi C_\delta S_{\lambda+\theta}\right)\dot{\delta} - n_{31}\left(C_\phi S_\delta C_{\lambda+\theta}\right)\dot{\theta} - \dot{n}_{31}\left(C_\phi S_\delta S_{\lambda+\theta} + S_\phi C_\delta\right)$$

$$\left(\dot{\tilde{C}}_{2C}\right)_{12} = \dot{n}_{33} C_\delta - \left(n_{33} S_\delta\right)\dot{\delta} \qquad \left(\dot{\tilde{C}}_{2C}\right)_{22} = -\dot{n}_{33} S_\delta - \left(n_{33} C_\delta\right)\dot{\delta} \qquad \left(\tilde{C}_{2C}\right)_{32} = -\dot{n}_{31} C_\delta + \left(n_{31} S_\delta\right)\dot{\delta}$$

Note in the above expressions that the value of the ***dependent speed*** $\dot{\theta}$ can be found from the independent speeds using the constraint equations.

Partial Angular Velocities:

As noted above, the B-frame components of the angular velocity of B can be written as

$$\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix}_B = \begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\theta}_F\end{Bmatrix}$$

The ***dependent speeds*** can now be ***eliminated*** from this expression using the constraint matrix $\left[J\right]$.

$$\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix}_B = \begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix}\left[J\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} \triangleq \left[WB\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix}$$

with

$$\left[WB\right] \triangleq \begin{bmatrix}C_\theta & 0 & 0\\ 0 & 0 & 0\\ S_\theta & 0 & 0\end{bmatrix} + \begin{bmatrix}-C_\phi S_\theta & 0 & 0\\ S_\phi & 1 & 0\\ C_\phi C_\theta & 0 & 0\end{bmatrix}\left[J\right]$$

The components of the partial angular velocities associated with B are in the columns of $\left[WB\right]$.

$$\frac{\partial}{\partial\dot{\phi}}\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix}_B = \begin{Bmatrix}WB_{11}\\ WB_{21}\\ WB_{31}\end{Bmatrix} \qquad \frac{\partial}{\partial\dot{\theta}_R}\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix}_B = \begin{Bmatrix}WB_{12}\\ WB_{22}\\ WB_{32}\end{Bmatrix} \qquad \frac{\partial}{\partial\dot{\delta}}\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix}_B = \begin{Bmatrix}WB_{13}\\ WB_{23}\\ WB_{33}\end{Bmatrix}$$

The B-frame components of the angular velocity of rear wheel A are written as

$$\left\{\begin{matrix}\omega_{A1}\\ \omega_{A2}\\ \omega_{A3}\end{matrix}\right\}_B = \left\{\begin{matrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{matrix}\right\}_B + \left\{\begin{matrix}0\\ \dot{\theta}_R\\ 0\end{matrix}\right\} = [WB]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} + \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} \triangleq [WA]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\}$$

with

$$[WA] = [WB] + \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

As for B, the partial angular velocity vectors of A are the columns of $[WA]$.

The C-frame components of the angular velocity of C can be written as

$$\left\{\begin{matrix}\omega_{C1}\\ \omega_{C2}\\ \omega_{C3}\end{matrix}\right\}_C = \left[R_{B2C}\right]\left\{\begin{matrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{matrix}\right\}_B + \left\{\begin{matrix}0\\ 0\\ \dot{\delta}\end{matrix}\right\} = \left[R_{B2C}\right][WB]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} \triangleq [WC]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\}$$

with

$$[WC] = \left[R_{B2C}\right][WB] + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

The partial angular velocity vectors of C are the columns of $[WC]$.

Finally, the C-frame components of the angular velocity of D can be written as

$$\begin{aligned}\left\{\begin{matrix}\omega_{D1}\\ \omega_{D2}\\ \omega_{D3}\end{matrix}\right\}_C &= \left[R_{B2C}\right]\left\{\begin{matrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{matrix}\right\}_B + \left\{\begin{matrix}0\\ \dot{\theta}_F\\ \dot{\delta}\end{matrix}\right\} = \left[R_{B2C}\right][WB]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} + \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\left\{\begin{matrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\theta}_F\end{matrix}\right\}\\ &= \left[R_{B2C}\right][WB]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} + \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[J]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\} \triangleq [WD]\left\{\begin{matrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{matrix}\right\}\end{aligned}$$

with

$$[WD] \triangleq \left[R_{B2C}\right][WB] + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[J] = [WC] + \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[J]$$

The partial angular velocity vectors of D are the columns of $[WD]$.

Partial Velocities:

Using results from above, the ***B-frame components*** of the ***velocity*** of G_A can now be written as

$$\begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_B = r_{RW}\begin{bmatrix} 0 & -C_\theta & 0 \\ 1 & 0 & 0 \\ 0 & -S_\theta & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + r_{RW}\begin{bmatrix} -C_\theta S_\phi & -C_\theta & 0 \\ 0 & 0 & 0 \\ -S_\theta S_\phi & -S_\theta & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

$$= r_{RW}\begin{bmatrix} 0 & -C_\theta & 0 \\ 1 & 0 & 0 \\ 0 & -S_\theta & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + r_{RW}\begin{bmatrix} -C_\theta S_\phi & -C_\theta & 0 \\ 0 & 0 & 0 \\ -S_\theta S_\phi & -S_\theta & 0 \end{bmatrix}[J]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} \triangleq [VA]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

with

$$[VA] = r_{RW}\begin{bmatrix} 0 & -C_\theta & 0 \\ 1 & 0 & 0 \\ 0 & -S_\theta & 0 \end{bmatrix} + r_{RW}\begin{bmatrix} -C_\theta S_\phi & -C_\theta & 0 \\ 0 & 0 & 0 \\ -S_\theta S_\phi & -S_\theta & 0 \end{bmatrix}[J]$$

The B-frame components of the partial velocity vectors of G_A are the columns of $[VA]$.

The B-frame components of the velocity of G_B can be written as

$$\begin{Bmatrix} v_{B1} \\ v_{B2} \\ v_{B3} \end{Bmatrix}_B = \begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_B + \begin{Bmatrix} v_{AB1} \\ v_{AB2} \\ v_{AB3} \end{Bmatrix}_B$$

Substituting previous results for the right-side velocity vectors gives

$$\begin{Bmatrix} v_{B1} \\ v_{B2} \\ v_{B3} \end{Bmatrix}_B = [VA]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ (x_B S_\theta + z_B C_\theta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -z_B S_\phi & -z_B & 0 \\ (x_B C_\theta - z_B S_\theta) C_\phi & 0 & 0 \\ -x_B S_\phi & -x_B & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix}$$

$$= \left[[VA] + \begin{bmatrix} 0 & 0 & 0 \\ (x_B S_\theta + z_B C_\theta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -z_B S_\phi & -z_B & 0 \\ (x_B C_\theta - z_B S_\theta) C_\phi & 0 & 0 \\ -x_B S_\phi & -x_B & 0 \end{bmatrix}[J]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

$$\triangleq [VB]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

with

$$[VB] = [VA] + \begin{bmatrix} 0 & 0 & 0 \\ (x_B S_\theta + z_B C_\theta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -z_B S_\phi & -z_B & 0 \\ (x_B C_\theta - z_B S_\theta) C_\phi & 0 & 0 \\ -x_B S_\phi & -x_B & 0 \end{bmatrix}[J]$$

Note the components of ${}^{R}\underset{\sim}{v}_{G_B/G_A}$ were found by replacing x_S and z_S with x_B and z_B in the expression for ${}^{R}\underset{\sim}{v}_{S/G_A}$. The B-frame components of the partial velocity vectors of B are the columns of $[VB]$.

The C-frame components of the velocity of G_C can be written as

$$\begin{Bmatrix} v_{C1} \\ v_{C2} \\ v_{C3} \end{Bmatrix}_C = \begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_C + \begin{Bmatrix} v_{AS1} \\ v_{AS2} \\ v_{AS3} \end{Bmatrix}_C + \begin{Bmatrix} v_{SC1} \\ v_{SC2} \\ v_{SC3} \end{Bmatrix}_C$$

Substituting previous results for the right-side velocity vectors and using the transformation matrix to convert components to the C-frame gives

$$\begin{aligned}
\begin{Bmatrix} v_{C1} \\ v_{C2} \\ v_{C3} \end{Bmatrix}_C &= [R_{B2C}][VA]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \\
&\quad [R_{B2C}]\begin{bmatrix} 0 & 0 & 0 \\ (x_S S_\theta + z_S C_\theta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + [R_{B2C}]\begin{bmatrix} -z_S S_\phi & -z_S & 0 \\ (x_S C_\theta - z_S S_\theta) C_\phi & 0 & 0 \\ -x_S S_\phi & -x_S & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix} + \\
&\quad \begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}\left[[R_{B2C}]\begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \\
&\quad \begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}[R_{B2C}]\begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\theta}_F \end{Bmatrix} \\
&= [R_{B2C}]\left[[VA] + \begin{bmatrix} 0 & 0 & 0 \\ x_S S_\theta + z_S C_\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -z_S S_\phi & -z_S & 0 \\ (x_S C_\theta - z_S S_\theta) C_\phi & 0 & 0 \\ -x_S S_\phi & -x_S & 0 \end{bmatrix}[J] \right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \\
&\quad \begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}\left[[R_{B2C}]\begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + [R_{B2C}]\begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}[J] \right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} \\
&\triangleq [VS]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \\
&\quad \begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}\left[[R_{B2C}]\begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + [R_{B2C}]\begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}[J] \right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}
\end{aligned}$$

Or, more simply,

$$\begin{Bmatrix} v_{C1} \\ v_{C2} \\ v_{C3} \end{Bmatrix}_C \triangleq [VC] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

Here,

$$[VC] = [VS] + \begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix} \left[[R_{B2C}] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + [R_{B2C}] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} [J] \right]$$

$$[VS] = [R_{B2C}] \left[[VA] + \begin{bmatrix} 0 & 0 & 0 \\ x_S S_\theta + z_S C_\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -z_S S_\phi & -z_S & 0 \\ (x_S C_\theta - z_S S_\theta) C_\phi & 0 & 0 \\ -x_S S_\phi & -x_S & 0 \end{bmatrix} [J] \right]$$

The C-frame components of the partial velocity vectors of G_C are the columns of $[VC]$. Note that the columns of $[VS]$ are the C-frame components of the partial velocity vectors of S.

Using a similar approach for the C-frame components of the velocity of G_D gives

$$\begin{Bmatrix} v_{D1} \\ v_{D2} \\ v_{D3} \end{Bmatrix}_C = \begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_C + \begin{Bmatrix} v_{AS1} \\ v_{AS2} \\ v_{AS3} \end{Bmatrix}_C + \begin{Bmatrix} v_{SD1} \\ v_{SD2} \\ v_{SD3} \end{Bmatrix}_C = [VS] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} +$$
$$\begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} \left[[R_{B2C}] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + [R_{B2C}] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} [J] \right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$
$$\triangleq [VD] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix}$$

with

$$[VD] = [VS] + \begin{bmatrix} 0 & z_D & 0 \\ -z_D & 0 & x_D \\ 0 & -x_D & 0 \end{bmatrix} \left[[R_{B2C}] \begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + [R_{B2C}] \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix} [J] \right]$$

The D-frame components of the partial velocity vectors of G_D are the columns of $[VD]$.

Angular Accelerations:

The angular acceleration of B can be written as

$$ {}^R\underset{\sim}{\alpha}_B = \frac{{}^R d\, {}^R\underset{\sim}{\omega}_B}{dt} = \frac{{}^B d\, {}^R\underset{\sim}{\omega}_B}{dt} = \dot{\omega}_{B1}\underset{\sim}{b}_1 + \dot{\omega}_{B2}\underset{\sim}{b}_2 + \dot{\omega}_{B3}\underset{\sim}{b}_3 \triangleq \alpha_{B1}\underset{\sim}{b}_1 + \alpha_{B2}\underset{\sim}{b}_2 + \alpha_{B3}\underset{\sim}{b}_3 $$

Differentiating previous results, the B-frame angular acceleration components of B can be written as

$$ \begin{Bmatrix} \alpha_{B1} \\ \alpha_{B2} \\ \alpha_{B3} \end{Bmatrix}_B = \begin{Bmatrix} \dot{\omega}_{B1} \\ \dot{\omega}_{B2} \\ \dot{\omega}_{B3} \end{Bmatrix}_B = [WB]\begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + \left[\dot{W}B\right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} $$

with

$$ \left[\dot{W}B\right] = \frac{d}{dt}[WB] = \begin{bmatrix} \left(-S_\theta\right)\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \\ \left(C_\theta\right)\dot{\theta} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \left(S_\phi S_\theta\right)\dot{\phi} - \left(C_\phi C_\theta\right)\dot{\theta} & 0 & 0 \\ \left(C_\phi\right)\dot{\phi} & 0 & 0 \\ \left(-S_\phi C_\theta\right)\dot{\phi} - \left(C_\phi S_\theta\right)\dot{\theta} & 0 & 0 \end{bmatrix}[J] + \begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}\left[\dot{J}\right] $$

The angular acceleration of A can be written as

$$ {}^R\underset{\sim}{\alpha}_A = \frac{{}^R d\, {}^R\underset{\sim}{\omega}_A}{dt} = \frac{{}^B d\, {}^R\underset{\sim}{\omega}_A}{dt} + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{\omega}_A\right) = \frac{{}^B d\, {}^R\underset{\sim}{\omega}_A}{dt} + \left({}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B + {}^B\underset{\sim}{\omega}_A\right)\right) = \frac{{}^B d\, {}^R\underset{\sim}{\omega}_A}{dt} + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{\omega}_A\right) $$

$$ = \left(\dot{\omega}_{A1}\underset{\sim}{b}_1 + \dot{\omega}_{A2}\underset{\sim}{b}_2 + \dot{\omega}_{A3}\underset{\sim}{b}_3\right) + \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_{B1} & \omega_{B2} & \omega_{B3} \\ 0 & \dot{\theta}_R & 0 \end{vmatrix} $$

Or,

$$ {}^R\underset{\sim}{\alpha}_A = \left(\dot{\omega}_{A1}\underset{\sim}{b}_1 + \dot{\omega}_{A2}\underset{\sim}{b}_2 + \dot{\omega}_{A3}\underset{\sim}{b}_3\right) + \left[\left(-\omega_{B3}\dot{\theta}_R\right)\underset{\sim}{b}_1 + \left(\omega_{B1}\dot{\theta}_R\right)\underset{\sim}{b}_3\right] \triangleq \alpha_{A1}\underset{\sim}{b}_1 + \alpha_{A2}\underset{\sim}{b}_2 + \alpha_{A3}\underset{\sim}{b}_3 $$

Differentiating previous results, the B-frame components of the angular acceleration of A can be written as

$$ \begin{Bmatrix} \alpha_{A1} \\ \alpha_{A2} \\ \alpha_{A3} \end{Bmatrix}_B = \begin{Bmatrix} \dot{\omega}_{A1} \\ \dot{\omega}_{A2} \\ \dot{\omega}_{A3} \end{Bmatrix}_B + \begin{Bmatrix} -\omega_{B3} \\ 0 \\ \omega_{B1} \end{Bmatrix}\dot{\theta}_R = [WA]\begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + \left[\dot{W}A\right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{Bmatrix} -\omega_{B3} \\ 0 \\ \omega_{B1} \end{Bmatrix}\dot{\theta}_R $$

with

$$ \left[\dot{W}A\right] = \frac{d}{dt}[WA] = \frac{d}{dt}\left[[WB] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right] = \left[\dot{W}B\right] $$

The angular acceleration of C can be written as

$$ {}^R\underset{\sim}{\alpha}_C = \frac{{}^R d\, {}^R\underset{\sim}{\omega}_C}{dt} = \frac{{}^C d\, {}^R\underset{\sim}{\omega}_C}{dt} = \dot{\omega}_{C1}\underset{\sim}{c}_1 + \dot{\omega}_{C2}\underset{\sim}{c}_2 + \dot{\omega}_{C3}\underset{\sim}{c}_3 \triangleq \alpha_{C1}\underset{\sim}{c}_1 + \alpha_{C2}\underset{\sim}{c}_2 + \alpha_{C3}\underset{\sim}{c}_3 $$

Again, differentiating previous results, the C-frame angular acceleration components of C can be written as

$$\begin{Bmatrix}\alpha_{C1}\\ \alpha_{C2}\\ \alpha_{C3}\end{Bmatrix}_C = \begin{Bmatrix}\dot{\omega}_{C1}\\ \dot{\omega}_{C2}\\ \dot{\omega}_{C3}\end{Bmatrix}_C = [WC]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}_R\\ \ddot{\delta}\end{Bmatrix} + [\dot{W}C]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix}$$

with

$$[\dot{W}C] = \frac{d}{dt}[WC] = \frac{d}{dt}\left[[R_{B2C}][WB] + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right] = [\dot{R}_{B2C}][WB] + [R_{B2C}][\dot{W}B]$$

$$[\dot{R}_{B2C}] = \frac{d}{dt}[R_{B2C}] = \begin{bmatrix}(-C_\lambda S_\delta)\dot{\delta} & (C_\delta)\dot{\delta} & (S_\lambda S_\delta)\dot{\delta}\\ (-C_\lambda C_\delta)\dot{\delta} & (-S_\delta)\dot{\delta} & (S_\lambda C_\delta)\dot{\delta}\\ 0&0&0\end{bmatrix} = \begin{bmatrix}(-C_\lambda S_\delta) & (C_\delta) & (S_\lambda S_\delta)\\ (-C_\lambda C_\delta) & (-S_\delta) & (S_\lambda C_\delta)\\ 0&0&0\end{bmatrix}\dot{\delta}$$

Finally, the angular acceleration of D can be written as

$$\begin{aligned}{}^R\underset{\sim}{\alpha}_D &= \frac{{}^R d\,{}^R\underset{\sim}{\omega}_D}{dt} = \frac{{}^C d\,{}^R\underset{\sim}{\omega}_D}{dt} + \left({}^R\underset{\sim}{\omega}_C \times {}^R\underset{\sim}{\omega}_D\right) = \frac{{}^C d\,{}^R\underset{\sim}{\omega}_D}{dt} + \left({}^R\underset{\sim}{\omega}_C \times \left({}^R\underset{\sim}{\omega}_C + {}^C\underset{\sim}{\omega}_D\right)\right) = \frac{{}^C d\,{}^R\underset{\sim}{\omega}_D}{dt} + \left({}^R\underset{\sim}{\omega}_C \times {}^C\underset{\sim}{\omega}_D\right)\\ &= \left(\dot{\omega}_{D1}\underset{\sim}{c}_1 + \dot{\omega}_{D2}\underset{\sim}{c}_2 + \dot{\omega}_{D3}\underset{\sim}{c}_3\right) + \begin{vmatrix}\underset{\sim}{c}_1 & \underset{\sim}{c}_2 & \underset{\sim}{c}_3\\ \omega_{C1} & \omega_{C2} & \omega_{C3}\\ 0 & \dot{\theta}_F & 0\end{vmatrix}\end{aligned}$$

Or,

$${}^R\underset{\sim}{\alpha}_D = \left(\dot{\omega}_{D1}\underset{\sim}{c}_1 + \dot{\omega}_{D2}\underset{\sim}{c}_2 + \dot{\omega}_{D3}\underset{\sim}{c}_3\right) + \left[\left(-\omega_{C3}\dot{\theta}_F\right)\underset{\sim}{c}_1 + \left(\omega_{C1}\dot{\theta}_F\right)\underset{\sim}{c}_3\right] \triangleq \alpha_{D1}\underset{\sim}{c}_1 + \alpha_{D2}\underset{\sim}{c}_2 + \alpha_{D3}\underset{\sim}{c}_3$$

The C-frame components of the angular acceleration of D can then be written as

$$\begin{Bmatrix}\alpha_{D1}\\ \alpha_{D2}\\ \alpha_{D3}\end{Bmatrix}_C = \begin{Bmatrix}\dot{\omega}_{D1}\\ \dot{\omega}_{D2}\\ \dot{\omega}_{D3}\end{Bmatrix}_C + \begin{Bmatrix}-\omega_{C3}\\ 0\\ \omega_{C1}\end{Bmatrix}\dot{\theta}_F = [WD]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}_R\\ \ddot{\delta}\end{Bmatrix} + [\dot{W}D]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix} + \begin{Bmatrix}-\omega_{C3}\\ 0\\ \omega_{C1}\end{Bmatrix}\dot{\theta}_F$$

with

$$[\dot{W}D] = \frac{d}{dt}\left[[R_{B2C}][WB] + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[J]\right] = [\dot{W}C] + \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[\dot{J}]$$

Mass-Center Accelerations:

The ***accelerations*** of the ***mass centers*** of the four bodies can be found by ***differentiating*** the ***velocities*** using the ***derivative rule***. As expected, the results for each body are ***linear*** in the ***second derivatives*** of the independent generalized coordinates. The expressions include the partial velocity matrices and their time derivatives. As before, skew symmetric matrices are used to perform the necessary vector cross products.

The accelerations of the mass centers of the bodies can be written as follows.

$$ {}^R\underset{\sim}{a}_{G_A} = \frac{{}^Rd}{dt}\left({}^R\underset{\sim}{v}_{G_A}\right) = \frac{{}^Bd}{dt}\left({}^R\underset{\sim}{v}_{G_A}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{G_A}\right) = \left(\dot{v}_{A1}\underset{\sim}{b}_1 + \dot{v}_{A2}\underset{\sim}{b}_2 + \dot{v}_{A3}\underset{\sim}{b}_3\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{G_A}\right) $$

$$ {}^R\underset{\sim}{a}_{G_B} = \frac{{}^Rd}{dt}\left({}^R\underset{\sim}{v}_{G_B}\right) = \frac{{}^Bd}{dt}\left({}^R\underset{\sim}{v}_{G_B}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{G_B}\right) = \left(\dot{v}_{B1}\underset{\sim}{b}_1 + \dot{v}_{B2}\underset{\sim}{b}_2 + \dot{v}_{B3}\underset{\sim}{b}_3\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{G_B}\right) $$

$$ {}^R\underset{\sim}{a}_{G_C} = \frac{{}^Rd}{dt}\left({}^R\underset{\sim}{v}_{G_C}\right) = \frac{{}^Cd}{dt}\left({}^R\underset{\sim}{v}_{G_C}\right) + \left({}^R\underset{\sim}{\omega}_C \times {}^R\underset{\sim}{v}_{G_C}\right) = \left(\dot{v}_{C1}\underset{\sim}{c}_1 + \dot{v}_{C2}\underset{\sim}{c}_2 + \dot{v}_{C3}\underset{\sim}{c}_3\right) + \left({}^R\underset{\sim}{\omega}_C \times {}^R\underset{\sim}{v}_{G_C}\right) $$

$$ {}^R\underset{\sim}{a}_{G_D} = \frac{{}^Rd}{dt}\left({}^R\underset{\sim}{v}_{G_D}\right) = \frac{{}^Cd}{dt}\left({}^R\underset{\sim}{v}_{G_D}\right) + \left({}^R\underset{\sim}{\omega}_C \times {}^R\underset{\sim}{v}_{G_D}\right) = \left(\dot{v}_{D1}\underset{\sim}{c}_1 + \dot{v}_{D2}\underset{\sim}{c}_2 + \dot{v}_{D3}\underset{\sim}{c}_3\right) + \left({}^R\underset{\sim}{\omega}_C \times {}^R\underset{\sim}{v}_{G_D}\right) $$

The B-frame components of ${}^R\underset{\sim}{a}_{G_A}$ can then be written as follows.

$$ \begin{Bmatrix} a_{A1} \\ a_{A2} \\ a_{A3} \end{Bmatrix}_B \triangleq \begin{Bmatrix} {}^R\underset{\sim}{a}_{G_A}\cdot\underset{\sim}{b}_1 \\ {}^R\underset{\sim}{a}_{G_A}\cdot\underset{\sim}{b}_2 \\ {}^R\underset{\sim}{a}_{G_A}\cdot\underset{\sim}{b}_3 \end{Bmatrix} = \begin{Bmatrix} \dot{v}_{A1} \\ \dot{v}_{A2} \\ \dot{v}_{A3} \end{Bmatrix}_B + \left[\tilde{\omega}_B\right]\begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix}_B = \left[VA\right]\begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + \left[\dot{VA}\right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_{B3} & \omega_{B2} \\ \omega_{B3} & 0 & -\omega_{B1} \\ -\omega_{B2} & \omega_{B1} & 0 \end{bmatrix}\begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix} $$

Here,

$$ \left[\dot{VA}\right] = r_{RW}\begin{bmatrix} 0 & \left(S_\theta\right)\dot{\theta} & 0 \\ 0 & 0 & 0 \\ 0 & \left(-C_\theta\right)\dot{\theta} & 0 \end{bmatrix} + r_{RW}\begin{bmatrix} -C_\theta S_\phi & -C_\theta & 0 \\ 0 & 0 & 0 \\ -S_\theta S_\phi & -S_\theta & 0 \end{bmatrix}\left[\dot{J}\right] + r_{RW}\begin{bmatrix} \left(S_\theta S_\phi\right)\dot{\theta} - \left(C_\theta C_\phi\right)\dot{\phi} & \left(S_\theta\right)\dot{\theta} & 0 \\ 0 & 0 & 0 \\ -\left(\left(C_\theta S_\phi\right)\dot{\theta} + \left(S_\theta C_\phi\right)\dot{\phi}\right) & \left(-C_\theta\right)\dot{\theta} & 0 \end{bmatrix}\left[J\right] $$

The B-frame components of ${}^R\underset{\sim}{a}_{G_B}$ can be written as follows.

$$ \begin{Bmatrix} a_{B1} \\ a_{B2} \\ a_{B3} \end{Bmatrix}_B \triangleq \begin{Bmatrix} {}^R\underset{\sim}{a}_{G_B}\cdot\underset{\sim}{b}_1 \\ {}^R\underset{\sim}{a}_{G_B}\cdot\underset{\sim}{b}_2 \\ {}^R\underset{\sim}{a}_{G_B}\cdot\underset{\sim}{b}_3 \end{Bmatrix} = \begin{Bmatrix} \dot{v}_{B1} \\ \dot{v}_{B2} \\ \dot{v}_{B3} \end{Bmatrix}_B + \left[\tilde{\omega}_B\right]\begin{Bmatrix} v_{B1} \\ v_{B2} \\ v_{B3} \end{Bmatrix}_B = \left[VB\right]\begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + \left[\dot{VB}\right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_{B3} & \omega_{B2} \\ \omega_{B3} & 0 & -\omega_{B1} \\ -\omega_{B2} & \omega_{B1} & 0 \end{bmatrix}\begin{Bmatrix} v_{B1} \\ v_{B2} \\ v_{B3} \end{Bmatrix} $$

Here,

$$
\left[\dot{V}B\right]=\left[\dot{V}A\right]+\begin{bmatrix} 0 & 0 & 0 \\ \left(x_B C_\theta - z_B S_\theta\right)\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} -\left(z_B C_\phi\right)\dot{\phi} & 0 & 0 \\ -C_\phi\left(x_B S_\theta + z_B C_\theta\right)\dot{\theta} - S_\phi\left(x_B C_\theta - z_B S_\theta\right)\dot{\phi} & 0 & 0 \\ -\left(x_B C_\phi\right)\dot{\phi} & 0 & 0 \end{bmatrix}\left[J\right]+
$$

$$
\begin{bmatrix} -z_B S_\phi & -z_B & 0 \\ \left(x_B C_\theta - z_B S_\theta\right)C_\phi & 0 & 0 \\ -x_B S_\phi & -x_B & 0 \end{bmatrix}\left[\dot{J}\right]
$$

The C-frame components of ${}^{R}\underset{\sim}{a}_{G_C}$ can be written as

$$
\begin{Bmatrix} a_{C1} \\ a_{C2} \\ a_{C3} \end{Bmatrix}_C \triangleq \begin{Bmatrix} {}^{R}\underset{\sim}{a}_{G_C}\cdot\underset{\sim}{c}_1 \\ {}^{R}\underset{\sim}{a}_{G_C}\cdot\underset{\sim}{c}_2 \\ {}^{R}\underset{\sim}{a}_{G_C}\cdot\underset{\sim}{c}_3 \end{Bmatrix} = \begin{Bmatrix} \dot{v}_{C1} \\ \dot{v}_{C2} \\ \dot{v}_{C3} \end{Bmatrix}_C + \left[\tilde{\omega}_C\right]\begin{Bmatrix} v_{C1} \\ v_{C2} \\ v_{C3} \end{Bmatrix}_C = \left[VC\right]\begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + \left[\dot{V}C\right]\begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_{C3} & \omega_{C2} \\ \omega_{C3} & 0 & -\omega_{C1} \\ -\omega_{C2} & \omega_{C1} & 0 \end{bmatrix}\begin{Bmatrix} v_{C1} \\ v_{C2} \\ v_{C3} \end{Bmatrix}
$$

with

$$
\left[\dot{V}C\right]=\left[\dot{V}S\right]+
$$

$$
\begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}\left[\left[\dot{R}_{B2C}\right]\begin{bmatrix} C_\theta & 0 & 0 \\ 0 & 0 & 0 \\ S_\theta & 0 & 0 \end{bmatrix}+\left[R_{B2C}\right]\begin{bmatrix} \left(-S_\theta\right)\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \\ \left(C_\theta\right)\dot{\theta} & 0 & 0 \end{bmatrix}\right]+
$$

$$
\begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}\left[\left[\dot{R}_{B2C}\right]\begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}\left[J\right]+\left[R_{B2C}\right]\begin{bmatrix} -C_\phi S_\theta & 0 & 0 \\ S_\phi & 1 & 0 \\ C_\phi C_\theta & 0 & 0 \end{bmatrix}\left[\dot{J}\right]\right]
$$

$$
\begin{bmatrix} 0 & z_C & 0 \\ -z_C & 0 & x_C \\ 0 & -x_C & 0 \end{bmatrix}\left[\left[R_{B2C}\right]\begin{bmatrix} \left(S_\phi S_\theta\right)\dot{\phi} - \left(C_\phi C_\theta\right)\dot{\theta} & 0 & 0 \\ \left(C_\phi\right)\dot{\phi} & 0 & 0 \\ -\left(S_\phi C_\theta\right)\dot{\phi} - \left(C_\phi S_\theta\right)\dot{\theta} & 0 & 0 \end{bmatrix}\left[J\right]\right]
$$

$$
\left[\dot{VS}\right]=\left[\dot{R}_{B2C}\right]\left[[VA]+\begin{bmatrix}0&0&0\\x_S S_\theta+z_S C_\theta&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}-z_S S_\phi&-z_S&0\\\left(x_S C_\theta-z_S S_\theta\right)C_\phi&0&0\\-x_S S_\phi&-x_S&0\end{bmatrix}[J]\right]+
$$

$$
\left[R_{B2C}\right]\left[\left[\dot{VA}\right]+\begin{bmatrix}0&0&0\\\left(x_S C_\theta-z_S S_\theta\right)\dot{\theta}&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}-z_S S_\phi&-z_S&0\\\left(x_S C_\theta-z_S S_\theta\right)C_\phi&0&0\\-x_S S_\phi&-x_S&0\end{bmatrix}\left[\dot{J}\right]\right]+
$$

$$
\left[R_{B2C}\right]\left[\begin{bmatrix}\left(-z_S C_\phi\right)\dot{\phi}&0&0\\-C_\phi\left(x_S S_\theta+z_S C_\theta\right)\dot{\theta}-S_\phi\left(x_S C_\theta-z_S S_\theta\right)\dot{\phi}&0&0\\\left(-x_S C_\phi\right)\dot{\phi}&0&0\end{bmatrix}[J]\right]
$$

Finally, the C-frame components of ${}^R\underset{\sim}{a}_{G_D}$ can be written as

$$
\begin{Bmatrix}a_{D1}\\a_{D2}\\a_{D3}\end{Bmatrix}\triangleq\begin{Bmatrix}{}^R\underset{\sim}{a}_{G_D}\cdot\underset{\sim}{c}_1\\{}^R\underset{\sim}{a}_{G_D}\cdot\underset{\sim}{c}_2\\{}^R\underset{\sim}{a}_{G_D}\cdot\underset{\sim}{c}_3\end{Bmatrix}=\begin{Bmatrix}\dot{v}_{D1}\\\dot{v}_{D2}\\\dot{v}_{D3}\end{Bmatrix}+\left[\tilde{\omega}_C\right]\begin{Bmatrix}v_{D1}\\v_{D2}\\v_{D3}\end{Bmatrix}=[VD]\begin{Bmatrix}\ddot{\phi}\\\ddot{\theta}_R\\\ddot{\delta}\end{Bmatrix}+\left[\dot{VD}\right]\begin{Bmatrix}\dot{\phi}\\\dot{\theta}_R\\\dot{\delta}\end{Bmatrix}+\begin{bmatrix}0&-\omega_{C3}&\omega_{C2}\\\omega_{C3}&0&-\omega_{C1}\\-\omega_{C2}&\omega_{C1}&0\end{bmatrix}\begin{Bmatrix}v_{D1}\\v_{D2}\\v_{D3}\end{Bmatrix}
$$

with

$$
\left[\dot{VD}\right]=\left[\dot{VS}\right]+
$$

$$
\begin{bmatrix}0&z_D&0\\-z_D&0&x_D\\0&-x_D&0\end{bmatrix}\left[\left[\dot{R}_{B2C}\right]\begin{bmatrix}C_\theta&0&0\\0&0&0\\S_\theta&0&0\end{bmatrix}+\left[R_{B2C}\right]\begin{bmatrix}\left(-S_\theta\right)\dot{\theta}&0&0\\0&0&0\\\left(C_\theta\right)\dot{\theta}&0&0\end{bmatrix}\right]+
$$

$$
\begin{bmatrix}0&z_D&0\\-z_D&0&x_D\\0&-x_D&0\end{bmatrix}\left[\left[\dot{R}_{B2C}\right]\begin{bmatrix}-C_\phi S_\theta&0&0\\S_\phi&1&0\\C_\phi C_\theta&0&0\end{bmatrix}[J]+\left[R_{B2C}\right]\begin{bmatrix}-C_\phi S_\theta&0&0\\S_\phi&1&0\\C_\phi C_\theta&0&0\end{bmatrix}\left[\dot{J}\right]\right]
$$

$$
\begin{bmatrix}0&z_D&0\\-z_D&0&x_D\\0&-x_D&0\end{bmatrix}\left[\left[R_{B2C}\right]\begin{bmatrix}\left(S_\phi S_\theta\right)\dot{\phi}-\left(C_\phi C_\theta\right)\dot{\theta}&0&0\\\left(C_\phi\right)\dot{\phi}&0&0\\-\left(S_\phi C_\theta\right)\dot{\phi}-\left(C_\phi S_\theta\right)\dot{\theta}&0&0\end{bmatrix}[J]\right]
$$

Kane's Equations of Motion:

Left Side of Equations

Given that the columns of the partial velocity matrices are the components of the partial velocity vectors, the ***contributions*** of the ***mass center accelerations*** to the left side of Kane's equations can be written as

$$\sum_{i=1}^{4}\left(m_i \, {}^R\underset{\sim}{a}_{G_i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_i}}{\partial u_k} \right) = m_A \left[VA\right]^T \begin{Bmatrix} a_{A1} \\ a_{A2} \\ a_{A3} \end{Bmatrix} + m_B \left[VB\right]^T \begin{Bmatrix} a_{B1} \\ a_{B2} \\ a_{B3} \end{Bmatrix} + m_C \left[VC\right]^T \begin{Bmatrix} a_{C1} \\ a_{C2} \\ a_{C3} \end{Bmatrix} + m_D \left[VD\right]^T \begin{Bmatrix} a_{D1} \\ a_{D2} \\ a_{D3} \end{Bmatrix}$$

$$= m_A \left[VA\right]^T \left[VA\right] \begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + m_A \left[VA\right]^T \left(\left[\dot{V}A\right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \left[\tilde{\omega}_B\right] \begin{Bmatrix} v_{A1} \\ v_{A2} \\ v_{A3} \end{Bmatrix} \right) +$$

$$m_B \left[VB\right]^T \left[VB\right] \begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + m_B \left[VB\right]^T \left(\left[\dot{V}B\right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \left[\tilde{\omega}_B\right] \begin{Bmatrix} v_{B1} \\ v_{B2} \\ v_{B3} \end{Bmatrix} \right) +$$

$$m_C \left[VC\right]^T \left[VC\right] \begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + m_C \left[VC\right]^T \left(\left[\dot{V}C\right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \left[\tilde{\omega}_C\right] \begin{Bmatrix} v_{C1} \\ v_{C2} \\ v_{C3} \end{Bmatrix} \right) +$$

$$m_D \left[VD\right]^T \left[VD\right] \begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} + m_D \left[VD\right]^T \left(\left[\dot{V}D\right] \begin{Bmatrix} \dot{\phi} \\ \dot{\theta}_R \\ \dot{\delta} \end{Bmatrix} + \left[\tilde{\omega}_C\right] \begin{Bmatrix} v_{D1} \\ v_{D2} \\ v_{D3} \end{Bmatrix} \right)$$

Given that the columns of the partial angular velocity matrices are the components of the partial angular velocity vectors, the ***contributions*** of the ***angular accelerations*** to the left side of Kane's equations can be written as follows. Note that advantage is taken of the fact that, due to the symmetries of the front and rear wheels, the ***inertias*** of the ***rear wheel*** are ***constant*** in the *B*-frame and the ***inertias*** of the ***front wheel*** are ***constant*** in the *C*-frame.

$$\sum_{i=1}^{4}\left[\left(\underset{\sim}{I}_{G_i}\cdot {}^R\underset{\sim}{\alpha}_{B_i}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}$$

$$=\left[WA\right]^T\left[I_{G_A}\right]\begin{Bmatrix}\alpha_{A1}\\ \alpha_{A2}\\ \alpha_{A3}\end{Bmatrix}+\left[WB\right]^T\left[I_{G_B}\right]\begin{Bmatrix}\alpha_{B1}\\ \alpha_{B2}\\ \alpha_{B3}\end{Bmatrix}+\left[WC\right]^T\left[I_{G_C}\right]\begin{Bmatrix}\alpha_{C1}\\ \alpha_{C2}\\ \alpha_{C3}\end{Bmatrix}+\left[WD\right]^T\left[I_{G_D}\right]\begin{Bmatrix}\alpha_{D1}\\ \alpha_{D2}\\ \alpha_{D3}\end{Bmatrix}$$

$$=\left[WA\right]^T\left[I_{G_A}\right]\left(\left[WA\right]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}_R\\ \ddot{\delta}\end{Bmatrix}+\left[\dot{W}A\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix}+\begin{Bmatrix}-\omega_{B3}\\ 0\\ \omega_{B1}\end{Bmatrix}\dot{\theta}_R\right)+\left[WB\right]^T\left[I_{G_B}\right]\left(\left[WB\right]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}_R\\ \ddot{\delta}\end{Bmatrix}+\left[\dot{W}B\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix}\right)+$$

$$\left[WC\right]^T\left[I_{G_C}\right]\left(\left[WC\right]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}_R\\ \ddot{\delta}\end{Bmatrix}+\left[\dot{W}C\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix}\right)+\left[WD\right]^T\left[I_{G_D}\right]\left(\left[WD\right]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}_R\\ \ddot{\delta}\end{Bmatrix}+\left[\dot{W}D\right]\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}_R\\ \dot{\delta}\end{Bmatrix}+\begin{Bmatrix}-\omega_{C3}\\ 0\\ \omega_{C1}\end{Bmatrix}\dot{\theta}_F\right)$$

Finally, the ***angular momentum terms*** on the left side of the equations can be written as follows.

$$\sum_{i=1}^{4}\left[\left({}^R\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_{B_i}}{\partial u_k}$$

$$=\left[WA\right]^T\left[\tilde{\omega}_A\right]\left[I_{G_A}\right]\begin{Bmatrix}\omega_{A1}\\ \omega_{A2}\\ \omega_{A3}\end{Bmatrix}+\left[WB\right]^T\left[\tilde{\omega}_B\right]\left[I_{G_B}\right]\begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix}+\left[WC\right]^T\left[\tilde{\omega}_C\right]\left[I_{G_C}\right]\begin{Bmatrix}\omega_{C1}\\ \omega_{C2}\\ \omega_{C3}\end{Bmatrix}$$

$$+\left[WD\right]^T\left[\tilde{\omega}_D\right]\left[I_{G_D}\right]\begin{Bmatrix}\omega_{D1}\\ \omega_{D2}\\ \omega_{D3}\end{Bmatrix}$$

Generalized Forces

Weight Forces: The contributions of the weight forces of the bodies can be written as

$$\left(F_{u_i}\right)_g=\left(\underset{\sim}{W}_A\cdot\frac{\partial\, {}^R\underset{\sim}{v}_{G_A}}{\partial u_i}\right)+\left(\underset{\sim}{W}_B\cdot\frac{\partial\, {}^R\underset{\sim}{v}_{G_B}}{\partial u_i}\right)+\left(\underset{\sim}{W}_C\cdot\frac{\partial\, {}^R\underset{\sim}{v}_{G_C}}{\partial u_i}\right)+\left(\underset{\sim}{W}_D\cdot\frac{\partial\, {}^R\underset{\sim}{v}_{G_D}}{\partial u_i}\right)\quad (i=1,2,3)$$

The components of the weight forces in the body frames can be written in matrix form as

$$\begin{Bmatrix}\underset{\sim}{W}_A\cdot\underset{\sim}{b}_1\\ \underset{\sim}{W}_A\cdot\underset{\sim}{b}_2\\ \underset{\sim}{W}_A\cdot\underset{\sim}{b}_3\end{Bmatrix}=\left[R_{R2B}\right]\begin{Bmatrix}0\\ 0\\ m_A g\end{Bmatrix}\qquad \begin{Bmatrix}\underset{\sim}{W}_B\cdot\underset{\sim}{b}_1\\ \underset{\sim}{W}_B\cdot\underset{\sim}{b}_2\\ \underset{\sim}{W}_B\cdot\underset{\sim}{b}_3\end{Bmatrix}=\left[R_{R2B}\right]\begin{Bmatrix}0\\ 0\\ m_B g\end{Bmatrix}$$

$$\begin{Bmatrix}\underset{\sim}{W}_C\cdot\underset{\sim}{c}_1\\ \underset{\sim}{W}_C\cdot\underset{\sim}{c}_2\\ \underset{\sim}{W}_C\cdot\underset{\sim}{c}_3\end{Bmatrix}=\left[R_{R2C}\right]\begin{Bmatrix}0\\ 0\\ m_C g\end{Bmatrix}\qquad \begin{Bmatrix}\underset{\sim}{W}_D\cdot\underset{\sim}{c}_1\\ \underset{\sim}{W}_D\cdot\underset{\sim}{c}_2\\ \underset{\sim}{W}_D\cdot\underset{\sim}{c}_3\end{Bmatrix}=\left[R_{R2C}\right]\begin{Bmatrix}0\\ 0\\ m_D g\end{Bmatrix}$$

The total contributions of the weight forces can then be written as

$$\{F_u\}_g = [VA]^T \left[R_{R2B}\right] \begin{Bmatrix} 0 \\ 0 \\ m_A g \end{Bmatrix} + [VB]^T \left[R_{R2B}\right] \begin{Bmatrix} 0 \\ 0 \\ m_B g \end{Bmatrix} + [VC]^T \left[R_{R2C}\right] \begin{Bmatrix} 0 \\ 0 \\ m_C g \end{Bmatrix} + [VD]^T \left[R_{R2C}\right] \begin{Bmatrix} 0 \\ 0 \\ m_D g \end{Bmatrix}$$

Driving Torque: Using the pedals, the rider can exert a torque $\underset{\sim}{T}_A = T_A \underset{\sim}{b}_2$ on the rear wheel. A reaction torque $\underset{\sim}{T}_B = -\underset{\sim}{T}_A = -T_A \underset{\sim}{b}_2$ is applied to the rear frame *B*. The contribution of these torques to the generalized forces can be written as

$$\left(F_{u_i}\right)_{T_A} = \left(T_A \underset{\sim}{b}_2\right) \cdot \left(\frac{\partial\, {}^R\underset{\sim}{\omega}_A}{\partial u_i}\right) + \left(-T_A \underset{\sim}{b}_2\right) \cdot \left(\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial u_i}\right) = \left(T_A \underset{\sim}{b}_2\right) \cdot \frac{\partial}{\partial u_i}\left({}^R\underset{\sim}{\omega}_A - {}^R\underset{\sim}{\omega}_B\right) = \left(T_A \underset{\sim}{b}_2\right) \cdot \frac{\partial}{\partial u_i}\left(\dot{\theta}_R \underset{\sim}{b}_2\right)$$

Or,

$$\{F_u\}_{T_A} = \begin{Bmatrix} 0 \\ T_A \\ 0 \end{Bmatrix}$$

Final Equations

Combining the results for the left and right sides of Kane's equations gives

$$[M]\{\dot{u}_I\} = [M] \begin{Bmatrix} \ddot{\phi} \\ \ddot{\theta}_R \\ \ddot{\delta} \end{Bmatrix} = \{f\}$$

where $[M]$, the ***generalized mass matrix***, is

$$\begin{aligned} [M] = {} & m_A [VA]^T [VA] + m_B [VB]^T [VB] + m_C [VC]^T [VC] + m_D [VD]^T [VD] \\ & + [WA]^T \left[I_{G_A}\right][WA] + [WB]^T \left[I_{G_B}\right][WB] + [WC]^T \left[I_{G_C}\right][WC] + [WD]^T \left[I_{G_D}\right][WD] \end{aligned}$$

and the vector on the right-side of the equation is

$$\{f\} = \{F_u\}_g + \{F_u\}_{T_A} - \left[m_A [VA]^T [\dot{V}A] + m_B [VB]^T [\dot{V}B] + m_C [VC]^T [\dot{V}C] + m_D [VD]^T [\dot{V}D] \right] \{u_I\}$$

$$- \left[[WA]^T [I_{G_A}][\dot{W}A] + [WB][I_{G_B}][\dot{W}B] + [WC]^T [I_{G_C}][\dot{W}C] + [WD]^T [I_{G_D}][\dot{W}D] \right] \{u_I\}$$

$$- m_A [VA]^T [\tilde{\omega}_B]\{v_A\} - m_B [VB]^T [\tilde{\omega}_B]\{v_B\} - m_C [VC]^T [\tilde{\omega}_C]\{v_C\} - m_D [VD]^T [\tilde{\omega}_C]\{v_D\}$$

$$-[WA]^T [I_{G_A}] \begin{Bmatrix} -\omega_{B3} \\ 0 \\ \omega_{B1} \end{Bmatrix} \dot{\theta}_R - [WD]^T [I_{G_D}] \begin{Bmatrix} -\omega_{C3} \\ 0 \\ \omega_{C1} \end{Bmatrix} \dot{\theta}_F - [WA]^T [\tilde{\omega}_A][I_{G_A}] \begin{Bmatrix} \omega_{A1} \\ \omega_{A2} \\ \omega_{A3} \end{Bmatrix}$$

$$-[WB]^T [\tilde{\omega}_B][I_{G_B}] \begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} - [WC]^T [\tilde{\omega}_C][I_{G_C}] \begin{Bmatrix} \omega_{C1} \\ \omega_{C2} \\ \omega_{C3} \end{Bmatrix} - [WD]^T [\tilde{\omega}_D][I_{G_D}] \begin{Bmatrix} \omega_{D1} \\ \omega_{D2} \\ \omega_{D3} \end{Bmatrix}$$

Final Equations of Motion:

Three Kane's Equations:

$$\{\dot{u}_I\} = [M]^{-1}\{f\}$$

Three Kinematical Constraint Equations:

$$\{\dot{u}_D\} = [J]\{\dot{u}_I\} + [\dot{J}]\{u_I\}$$

Six Kinematical Definitions:

$$\{u_I\} = [\dot{\phi} \quad \dot{\theta}_R \quad \dot{\delta}]^T \quad \text{and} \quad \{u_D\} = [\dot{\psi} \quad \dot{\theta} \quad \dot{\theta}_F]^T$$

These represent ***twelve***, ***first-order***, ***ordinary differential equations***. Given ***initial values*** for the six generalized coordinates $(\phi, \theta_R, \delta, \psi, \theta, \theta_F)$ and their time derivatives $(\dot{\phi}, \dot{\theta}_R, \dot{\delta}, \dot{\psi}, \dot{\theta}, \dot{\theta}_F)$, these equations can be ***numerically integrated*** to find their ***future values***.

Initial Conditions of the Generalized Coordinates:

It is assumed that the roll angle ϕ, the rear wheel angle θ_R, and the steering angle δ form a set of three independent generalized coordinates. As independent coordinates, their ***initial values*** can be ***specified independently***. The values of the rest of the coordinates must be chosen to be ***consistent*** with the ***rolling constraints***.

The ***distance*** of the front wheel contact point Q from the surface is ***independent*** of the values of the x and y coordinates of the rear contact point P, the ***yaw angle*** of the bicycle (ψ), and the ***angle of the front wheel*** relative to the front frame (θ_F). Hence, their ***initial values*** can be ***arbitrarily specified***. For ***simplicity***, their ***initial values*** are all taken to be ***zero***. That leaves the ***initial value*** of the ***rear frame pitch*** angle θ to be determined.

To find the pitch angle, the ***z coordinate*** of $\underset{\sim}{r}_{Q/P}$ the position vector of Q relative to P is set to ***zero***. This provides a ***single equation*** to solve for the initial value of θ. To this end, $\underset{\sim}{r}_{Q/P}$ can be written as follows.

$$\underset{\sim}{r}_{Q/P} = \underset{\sim}{r}_{G_A/P} + \underset{\sim}{r}_{S/G_A} + \underset{\sim}{r}_{G_D/S} + \underset{\sim}{r}_{Q/G_D}$$

Here,

$$\underset{\sim}{r}_{G_A/P} = r_{RW}\left(S_\theta \underset{\sim}{b}_1 - C_\theta \underset{\sim}{b}_3\right) \qquad \underset{\sim}{r}_{S/G_A} = x_S \underset{\sim}{b}_1 - z_S \underset{\sim}{b}_3 \qquad \underset{\sim}{r}_{G_D/S} = x_D \underset{\sim}{c}_1 + z_D \underset{\sim}{c}_3$$

$$\underset{\sim}{r}_{Q/G_D} = \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\left(n_{31} \underset{\sim}{c}_1 + n_{33} \underset{\sim}{c}_3\right)$$

The components of each of these vectors can be resolved in the inertial frame R using transformation matrices. In matrix form,

$$\begin{Bmatrix} \underset{\sim}{r}_{Q/P} \cdot \underset{\sim}{N}_1 \\ \underset{\sim}{r}_{Q/P} \cdot \underset{\sim}{N}_2 \\ \underset{\sim}{r}_{Q/P} \cdot \underset{\sim}{N}_3 \end{Bmatrix} = \left[R_{B2R}\right]\begin{Bmatrix} r_{RW} S_\theta + x_S \\ 0 \\ -r_{RW} C_\theta - z_S \end{Bmatrix} + \left[R_{C2R}\right]\left(\begin{Bmatrix} x_D \\ 0 \\ z_D \end{Bmatrix} + \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\begin{Bmatrix} n_{31} \\ 0 \\ n_{33} \end{Bmatrix}\right)$$

Given the initial values for the specified variables, the transformation matrices can be written as

$$\left[R_{B2R}\right] = \left[R_{R2B}\right]^T = \begin{bmatrix} C_\theta & S_\phi S_\theta & -C_\phi S_\theta \\ 0 & C_\phi & S_\phi \\ S_\theta & -S_\phi C_\theta & C_\phi C_\theta \end{bmatrix}^T = \begin{bmatrix} C_\theta & 0 & S_\theta \\ S_\phi S_\theta & C_\phi & -S_\phi C_\theta \\ -C_\phi S_\theta & S_\phi & C_\phi C_\theta \end{bmatrix}$$

$$\begin{aligned} \left[R_{C2R}\right] &= \left[R_{R2B}\right]^T \left[R_{B2C}\right]^T = \left[R_{B2R}\right]\left[R_{C2B}\right] \\ &= \begin{bmatrix} C_\theta & 0 & S_\theta \\ S_\phi S_\theta & C_\phi & -S_\phi C_\theta \\ -C_\phi S_\theta & S_\phi & C_\phi C_\theta \end{bmatrix}\begin{bmatrix} C_\lambda C_\delta & -C_\lambda S_\delta & S_\lambda \\ S_\delta & C_\delta & 0 \\ -S_\lambda C_\delta & S_\lambda S_\delta & C_\lambda \end{bmatrix} \triangleq \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \end{aligned}$$

Using these results, the $\underset{\sim}{N}_3$ component of $\underset{\sim}{r}_{Q/P}$ can be written as

$$\begin{aligned} \underset{\sim}{r}_{Q/P} \cdot \underset{\sim}{N}_3 &= \left[-C_\phi S_\theta \quad S_\phi \quad C_\phi C_\theta\right]\begin{Bmatrix} r_{RW} S_\theta + x_S \\ 0 \\ -r_{RW} C_\theta - z_S \end{Bmatrix} + \left[n_{31} \quad n_{32} \quad n_{33}\right]\left(\begin{Bmatrix} x_D \\ 0 \\ z_D \end{Bmatrix} + \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\begin{Bmatrix} n_{31} \\ 0 \\ n_{33} \end{Bmatrix}\right) \\ &= -C_\phi S_\theta\left(r_{RW} S_\theta + x_S\right) - C_\phi C_\theta\left(r_{RW} C_\theta + z_S\right) + x_D n_{31} + z_D n_{33} \\ &\quad + \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\left(n_{31}^2 + n_{33}^2\right) \end{aligned}$$

To simplify this expression, consider the following

$$\begin{aligned} -C_\phi S_\theta\left(r_{RW}S_\theta + x_S\right) - C_\phi C_\theta\left(r_{RW}C_\theta + z_S\right) &= -r_{RW}C_\phi\left(S_\theta^2 + C_\theta^2\right) - \left(x_S C_\phi\right)S_\theta - \left(z_S C_\phi\right)C_\theta \\ &= -\left(r_{RW} + x_S S_\theta + z_S C_\theta\right)C_\phi \end{aligned}$$

$$\begin{aligned} x_D n_{31} + z_D n_{33} &= x_D\left(-C_\phi S_\theta C_\lambda C_\delta + S_\phi S_\delta - C_\phi C_\theta S_\lambda C_\delta\right) + z_D\left(-C_\phi S_\theta S_\lambda + C_\phi C_\theta C_\lambda\right) \\ &= x_D S_\phi S_\delta - x_D C_\phi C_\delta\left(S_\theta C_\lambda + C_\theta S_\lambda\right) + z_D C_\phi\left(C_\theta C_\lambda - S_\theta S_\lambda\right) \\ &= x_D S_\phi S_\delta - x_D C_\phi C_\delta S_{\lambda+\theta} + z_D C_\phi C_{\lambda+\theta} \end{aligned}$$

$$\frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\left(n_{31}^2 + n_{33}^2\right) = r_{FW}\sqrt{n_{31}^2 + n_{33}^2}$$

$$= r_{FW}\left(\left(-C_\phi S_\theta C_\lambda C_\delta + S_\phi S_\delta - C_\phi C_\theta S_\lambda C_\delta\right)^2 + \left(-C_\phi S_\theta S_\lambda + C_\phi C_\theta C_\lambda\right)^2\right)^{\frac{1}{2}}$$

$$= r_{FW}\left(\left(+S_\phi S_\delta - C_\phi C_\delta\left(S_\theta C_\lambda + C_\theta S_\lambda\right)\right)^2 + C_\phi^2\left(C_\theta C_\lambda - S_\theta S_\lambda\right)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{r_{FW}}{\sqrt{n_{31}^2 + n_{33}^2}}\left(n_{31}^2 + n_{33}^2\right) = r_{FW}\left(\left(+S_\phi S_\delta - C_\phi C_\delta S_{\lambda+\theta}\right)^2 + C_\phi^2 C_{\lambda+\theta}^2\right)^{\frac{1}{2}}$$

So, given ***arbitrary values*** of ***roll angle*** ϕ and ***steering angle*** δ, the following ***non-linear, algebraic equation*** can be solved to find the pitch angle θ.

$$\begin{aligned} &-\left(r_{RW} + x_S S_\theta + z_S C_\theta\right)C_\phi + x_D S_\phi S_\delta - x_D C_\phi C_\delta S_{\lambda+\theta} + z_D C_\phi C_{\lambda+\theta} \\ &\quad + r_{FW}\left(\left(+S_\phi S_\delta - C_\phi C_\delta S_{\lambda+\theta}\right)^2 + C_\phi^2 C_{\lambda+\theta}^2\right)^{\frac{1}{2}} = 0 \end{aligned}$$

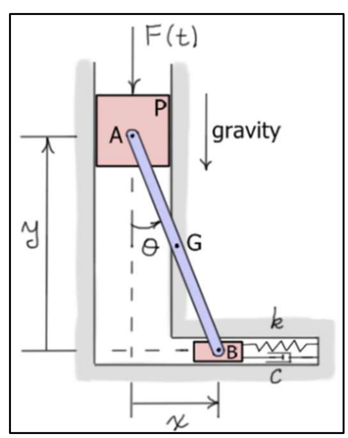
Note: It can be shown that the pitch angle is zero if the roll and steering angles are zero.

Numerical Results:

Numerical results for free motion of the bicycle are given in Unit 7 of this volume.

Exercises:

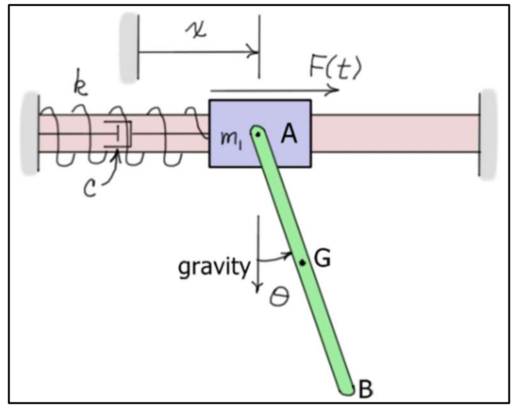
5.1 Find the differential equation of motion of the single degree-of-freedom system shown. The system consists of slender bar *AB* of mass *m* and length ℓ and a piston *P* of mass m_p. The system is driven by the force $F(t) = F_0 + F_1\sin(\omega t)$ and gravity. A spring and damper are attached to the light slider at *B*. The spring is unstretched when $x = 0$. Use d'Alembert's principle to find the equation of motion. Use the angle θ as the generalized coordinate. Neglect friction.



Answer:

$$\left[\tfrac{1}{3}m\ell^2 + m_p\ell^2 S_\theta^2\right]\ddot{\theta} + (m_p\ell^2 S_\theta C_\theta)\dot{\theta}^2 + (c\ell^2 C_\theta^2)\dot{\theta} + k\ell^2 S_\theta C_\theta - \left[m_p g + \tfrac{1}{2}mg + F(t)\right]\ell S_\theta = 0$$

5.2 Find the differential equations of motion of the two degree-of-freedom system shown. The system consists of a mass m_1 that translates along a fixed horizontal bar and a uniform slender bar AB that is pinned to m_1 at A. Bar AB has mass m_2 and length ℓ. Mass m_1 is attached to the fixed support by a spring of stiffness k and a linear viscous damper with coefficient c. The spring is unstretched when $x = 0$. The system is driven by gravity and the force $F(t) = F_0 \sin(\omega t)$ applied to m_1. Use d'Alembert's principle for find the equations of motion for this system. Use the variables x and θ as the generalized coordinates. Neglect friction.



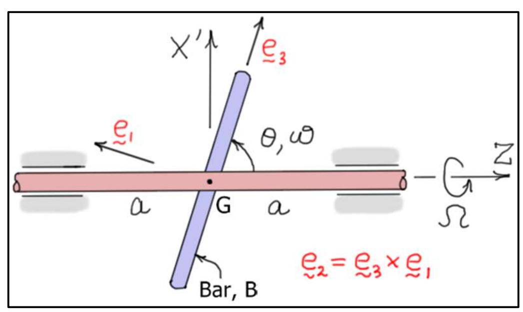
Answers:

$$\left(m_1 + m_2\right)\ddot{x} + (\tfrac{1}{2}m_2\ell C_\theta)\ddot{\theta} - (\tfrac{1}{2}m_2\ell S_\theta)\dot{\theta}^2 + c\dot{x} + kx = F(t)$$
$$\left(\tfrac{1}{2}m_2\ell C_\theta\right)\ddot{x} + \left(\tfrac{1}{3}m_2\ell^2\right)\ddot{\theta} + \tfrac{1}{2}m_2 g\ell S_\theta = 0$$

5.3 The two degree of freedom system consists of a uniform slender bar B of length ℓ and mass m that is pinned through the center of a shaft of mass m_s and radius r. The rotation of the shaft about the Z-axis is described by the angle $\phi\ (\dot{\phi} = \Omega)$, and the rotation of the bar B about the Y' axis is described by the angle $\theta\ (\dot{\theta} = \omega)$. A motor torque M_ϕ is applied to the shaft about the Z-axis, and a motor torque M_θ is applied to B by the shaft about the Y' axis. Using d'Alembert's principle, find the equations of motion of the system. Use the angles θ and ϕ as the generalized coordinates.

Answers:

$$\left[\tfrac{1}{2}m_s r^2 + \tfrac{1}{12}m\ell^2 S_\theta^2\right]\ddot{\phi} + \left(\tfrac{1}{6}m\ell^2 S_\theta C_\theta\right)\dot{\theta}\dot{\phi} = M_\phi$$
$$\left(\tfrac{1}{12}m\ell^2\right)\ddot{\theta} - \left(\tfrac{1}{12}m\ell^2 S_\theta C_\theta\right)\dot{\phi}^2 = M_\theta$$



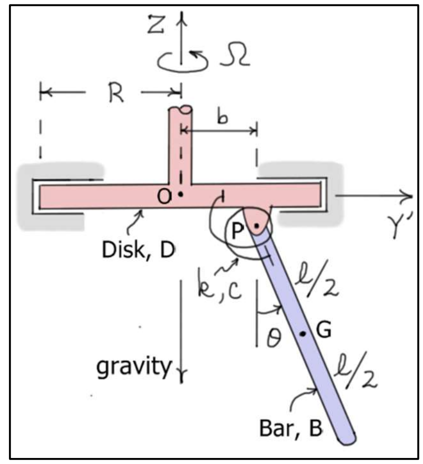
5.4 Find the equations of motion for the system of Exercise 5.3 using Kane's equations assuming the shaft is light. Use $(u_1, u_2) = (\omega_1, \omega_2) \triangleq ({}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{e}_1, {}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{e}_2)$ as the two independent generalized speeds.

Answers:

$$\left(m\ell^2/12\right)S_\theta\dot{\omega}_1 + \left(m\ell^2/12\right)C_\theta\omega_1\omega_2 = -M_\phi$$
$$\left(m\ell^2/12\right)\dot{\omega}_2 - \left(m\ell^2 C_\theta/12S_\theta\right)\omega_1^2 = M_\theta$$

$$\dot{\phi} = -\omega_1/S_\theta$$
$$\dot{\theta} = \omega_2$$

5.5 The two degree of freedom system consists of a disk D of mass m_d and radius R, and a uniform slender bar B of mass m and length ℓ. The rotation of the disk about the Z axis is described by the angle ϕ $(\dot{\phi} = \Omega)$, and the rotation of the bar B about the X' axis is described by the angle θ $(\dot{\theta} = \omega)$. A linear rotational spring-damper is located between B and D at the pin P. The spring has stiffness k and is unstretched when $\theta = 0$. The damper has coefficient c. A motor torque M_ϕ is applied to the disk about the Z axis, and a motor torque M_θ is applied to B by D about the X' axis. Use d'Alembert's principle to find the equations of motion of the system. Use the angles θ and ϕ as the generalized coordinates.



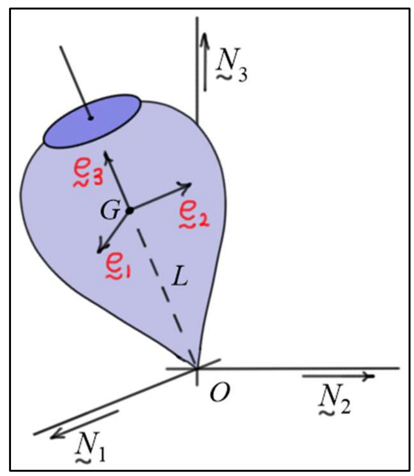
Answers:

$$\left[\tfrac{1}{2}m_d R^2 + m\left(b + \tfrac{1}{2}\ell S_\theta\right)^2 + \tfrac{1}{12}m\ell^2 S_\theta^2\right]\ddot{\phi} + (mb\ell C_\theta + \tfrac{2}{3}m\ell^2 S_\theta C_\theta)\dot{\theta}\dot{\phi} = M_\phi$$

$$\left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta} - \left[\tfrac{1}{2}mb\ell C_\theta + \tfrac{1}{3}m\ell^2 S_\theta C_\theta\right]\dot{\phi}^2 + c\dot{\theta} + k\theta + \tfrac{1}{2}mg\ell S_\theta = M_\theta$$

5.6 Spinning Top

Using Kane's equations, find the equations of motion of the ***three*** degree-of freedom spinning top shown in the diagram. Assume the moments of inertia of the top about the $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ directions are $I_1 = I_2 = I$, and the moment of inertia about the $\underset{\sim}{e}_3$ direction is I_3. Also, assume point O is ***fixed*** and acts like a ball in socket joint. Use ***Euler parameters*** to define the orientation of the top and define the generalized speeds to be $(u_1, u_2, u_3) = (\omega_1, \omega_2, \omega_3)$ the body-fixed angular velocity components, where $\omega_i = {}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{e}_i$ $(i = 1,2,3)$. The unit vector set $T : (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is fixed in and rotates with the top.

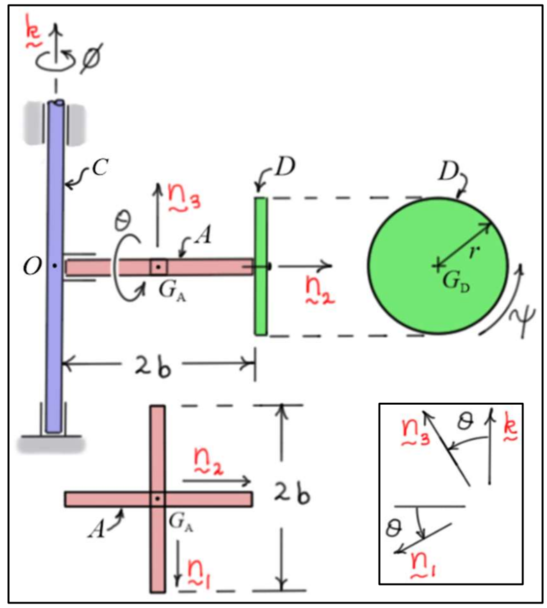


Answers:

$$\left(I + mL^2\right)\dot{\omega}_1 + \left(I_3 - I - mL^2\right)\omega_2\omega_3 = 2mgL\left(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4\right)$$
$$\left(I + mL^2\right)\dot{\omega}_2 + \left(I + mL^2 - I_3\right)\omega_1\omega_3 = 2mgL\left(\varepsilon_2\varepsilon_4 - \varepsilon_1\varepsilon_3\right)$$
$$\omega_3 = \text{constant}$$

$$\begin{Bmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\\ \dot{\varepsilon}_3\\ \dot{\varepsilon}_4\end{Bmatrix} = \frac{1}{2}\begin{bmatrix}\varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1\\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2\\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3\\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4\end{bmatrix}\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ 0\end{Bmatrix}$$

5.7 The system shown consists of three bodies: the column C, the cross-shaped frame A and the disk D. C is connected to the ground by a revolute joint allowing motion about a fixed vertical axis. A is connected to C and D is connected to A by revolute joints each allowing rotation about the rotating $\underset{\sim}{n}_2$ direction. The three relative rotations are described by the angles ϕ, θ and ψ, respectively. The system is driven by three known torques T_ϕ, T_θ and T_ψ located at each of the revolute joints. The unit vector set $A:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ is fixed in frame A. The points G_A and G_D represent the mass centers of A and D. The mass center of the system G is located a distance d_A to the right of G_A and a distance d_D to the left of G_D.



Using Kane's equations, find the equations of motion for the system. Use $(u_1, u_2, u_3) = (\omega_1, \omega_2, \omega_D) \triangleq \left({}^R\underset{\sim}{\omega}_A \cdot \underset{\sim}{n}_1, {}^R\underset{\sim}{\omega}_A \cdot \underset{\sim}{n}_2, {}^A\underset{\sim}{\omega}_D \cdot \underset{\sim}{n}_2\right)$ as the three independent generalized speeds.

Answers: (7 Unknowns: $\omega_1, \omega_2, \omega_3, \omega_D, \phi, \theta, \psi$)

3 Kane's equations of motion (after some manipulation of the first equation)

$$-\left[I_{11}^{G_A} + m_A b^2 + I_{11}^{G_D} + 4m_D b^2\right] S_\theta \dot{\omega}_1 + \left[I_{33}^{G_A} + m_A b^2 + I_{33}^{G_D} + 4m_D b^2\right] C_\theta \dot{\omega}_3$$
$$-\left[I_{11}^{G_A} + m_A b^2 + I_{11}^{G_D} + 4m_D b^2\right] C_\theta \omega_1 \omega_2 - \left[I_{33}^{G_A} + m_A b^2 + I_{33}^{G_D} + 4m_D b^2\right] S_\theta \omega_2 \omega_3 = T_\phi$$

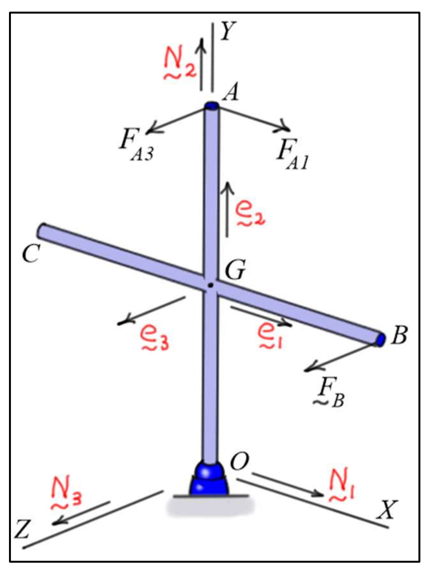
$$\left(I_{22}^{G_A} + I_{22}^{G_D}\right)\dot{\omega}_2 + I_{22}^{G_D}\dot{\omega}_D + \left(I_{11}^{G_A} - I_{33}^{G_A}\right)\omega_1\omega_3 = T_\theta$$

$$I_{22}^{G_D}\left(\dot{\omega}_2 + \dot{\omega}_D\right) = T_\psi$$

4 kinematic equations

$\omega_1 = -\dot{\phi} S_\theta$ $\quad$ $\omega_2 = \dot{\theta}$ $\quad$ $\omega_3 = \dot{\phi} C_\theta$ $\quad$ $\omega_D = \dot{\psi}$

5.8 The bracket $OABC$ shown in the diagram (shaped like a "+" sign) is attached to the ground with a ball-and-socket joint at O. The bars OA and BC are identical slender bars with mass m and length L. The orientation of the bracket is to be described using a 1-2-3 orientation angle sequence. In the configuration shown, all angles are ***zero*** so the ***inertial*** unit vectors $(\underset{\sim}{N}_i,\ i = 1,2,3)$ are aligned with the ***body-fixed*** unit vectors $(\underset{\sim}{e}_i,\ i = 1,2,3)$. The bracket moves under the action of its own weight at G and external forces at A and B. The forces at A and B can be written as $\underset{\sim}{F}_A = F_{A1}\,\underset{\sim}{e}_1 + F_{A3}\,\underset{\sim}{e}_3$ and $\underset{\sim}{F}_B = F_B\,\underset{\sim}{e}_3$. The weight force is $\underset{\sim}{W} = -2mg\,\underset{\sim}{N}_2$.



The configuration of the bracket is described by the three orientation angles $\{q_k\} = \{\theta_1, \theta_2, \theta_3\}$ and the body-fixed angular velocity components $\{u_k\} = \{\omega_1, \omega_2, \omega_3\}$. Find the ***three equations of motion*** of the bracket using Kane's equations using the body-fixed angular velocity components as the three independent generalized speeds.

Answers:

$$\left(\frac{7}{12}mL^2\right)\dot{\omega}_1 + \left(\frac{7}{12}mL^2\right)\omega_2\omega_3 = F_{\omega_1} = LF_{A3} + LF_B/2 + mgLS_1C_2$$
$$\left(\frac{1}{12}mL^2\right)\dot{\omega}_2 - \left(\frac{1}{12}mL^2\right)\omega_1\omega_3 = F_{\omega_2} = -LF_B/2$$
$$\left(\frac{2}{3}mL^2\right)\dot{\omega}_3 - \left(\frac{1}{2}mL^2\right)\omega_1\omega_2 = F_{\omega_3} = -LF_{A1} + mgL(C_1S_3 + S_1S_2C_3)$$

References:

1. L. Meirovitch, *Methods of Analytical Dynamics*, McGraw-Hill, 1970.
2. T.R. Kane, P.W. Likins, and D.A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983
3. T.R. Kane and D.A. Levinson, *Dynamics: Theory and Application*, McGraw-Hill, 1985
4. R.L. Huston, *Multibody Dynamics*, Butterworth-Heinemann, 1990
5. H. Baruh, *Analytical Dynamics*, McGraw-Hill, 1999
6. H. Josephs and R.L. Huston, *Dynamics of Mechanical Systems*, CRC Press, 2002
7. R.C. Hibbeler, *Engineering Mechanics: Dynamics*, 13th Ed., Pearson Prentice Hall, 2013
8. J.L. Meriam and L.G. Craig, *Engineering Mechanics: Dynamics*, 3rd Ed, 1992
9. F.P. Beer and E.R. Johnston, Jr. *Vector Mechanics for Engineers: Dynamics*, 4th Ed, 1984
10. Bicycle references:
    a. B.W. Kostich, "Development of Empirical and Virtual Tools for the Study of Bicycle Safety", MS Thesis, Western Michigan University, 2017
    b. J.K. Moore, "Human Control of a Bicycle", Ph.D. Dissertation, University of California-Davis, 2012
    c. J.P. Meijaard, J.M. Papadopoulos, A. Ruina, and A.L. Schwab, "Linearized Dynamics Equations for the Balance and Steer of a Bicycle: a Benchmark and Review", *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 463(2084), 1955–1982, 2007
    d. P. Basu-Mandal, A. Chatterjee, and J.M. Papadopoulos, "Hands-free Circular Motions of a Benchmark Bicycle. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 463(March), 1983–2003, 2007