An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 5

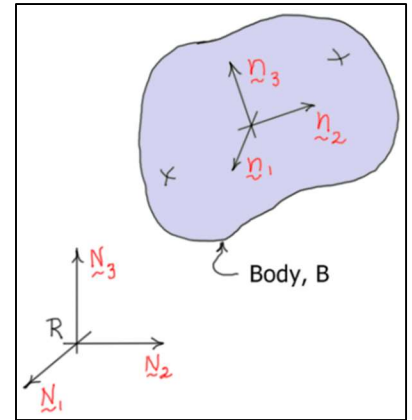# Rigid Body Orientation, Orientation Angles, and Angular Velocity

Summary

In Unit 1 the concepts of **angular velocity** and **angular acceleration** were introduced, and examples were provided to illustrate how to calculate angular velocity and angular acceleration vectors for mechanical systems in which components are connected by simple revolute (pin) joints. These concepts will now be **generalized** and applied to bodies in three-dimensional motion irrespective of how they may be connected (if at all) to other bodies or to the ground. Specifically, this unit describes how to use **sequences of angles** to describe the **orientation** and **angular motion** of rigid bodies in three dimensions.

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Orientation Angles of a Rigid Body in Three Dimensions

To describe the **general orientation** of a rigid body in three dimensions, consider the rigid body shown in the figure at the right. Here, there are two reference frames – the **base frame** $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$, and the **body-fixed frame** $B:\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$. In an arbitrary position, none of the unit vectors of the two frames are aligned. Generally speaking, there are **two methods** for describing the orientation of B relative to the base frame R.

The first (and most used) method of **orienting** a body in three dimensions involves the use of **orientation angles**. These are easy to **visualize**, but they are **not unique,** and they give rise to **mathematical singularities** in certain positions. The second method involves the use of **Euler** (or Euler-like) **parameters**. These are **not easy to visualize**; however, they are **unique,** and they have **no mathematical singularities**. The following sections discuss the use of **orientation angles** to describe angular position and motion of rigid bodies. **Euler parameters** are discussed in detail in **Unit 6**.

Simple Rotations

Simple rotations are defined as **right-handed** (or dextral) rotations about a single axis. For example, assume initially that the directions $\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$ are aligned with the directions $\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$. Then, an X-rotation is defined as a right-handed rotation of B about $\underset{\sim}{N}_1$ (or $\underset{\sim}{n}_1$), a Y-rotation as a right-handed rotation about $\underset{\sim}{N}_2$ (or $\underset{\sim}{n}_2$), and a Z-rotation as a right-handed rotation about $\underset{\sim}{N}_3$ (or $\underset{\sim}{n}_3$). For each of these simple rotations, the unit vectors of the two reference frames can be related using the following matrix equations.
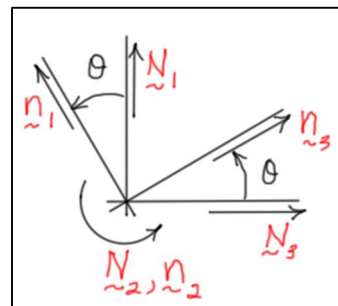
X-Rotation:
$$\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$
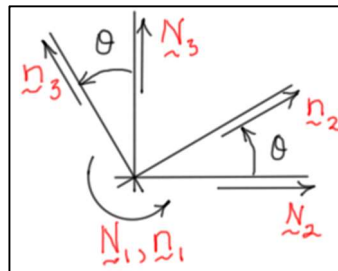
Y-Rotation:
$$\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

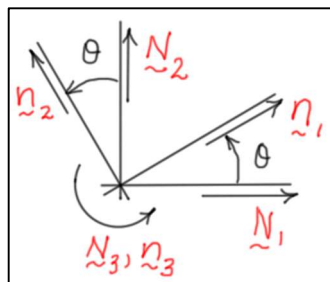$$\underline{Z\text{-Rotation:}} \quad \begin{Bmatrix} \underline{n}_1 \\ \underline{n}_2 \\ \underline{n}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underline{N}_1 \\ \underline{N}_2 \\ \underline{N}_3 \end{Bmatrix}$$


In each of the above equations, S_θ and C_θ represent the sine and cosine of the angle θ.

The coefficient matrices in the above equations are called "***transformation***" or "***rotation***" matrices. They are ***orthogonal*** matrices with a ***determinant*** of +1. As with all orthogonal matrices, the ***inverses*** of these matrices are simply their ***transposes***. Hence, it is easy to ***invert*** the equations to express the base system unit vectors in terms of the body-fixed unit vectors.

General Orientations

A rigid body can be moved into ***any orientation*** (relative to a base frame) using a ***sequence*** of three successive simple rotations. These rotations can occur about the ***base-frame axes*** or the ***body-frame axes***. One common example is a ***body-fixed***, ***1-2-3*** rotation sequence. Here, "1-2-3" has been used to stand for successive rotations about the $\underline{n}_1$, $\underline{n}_2$, $\underline{n}_3$ directions. To work through the rotations, ***intermediate reference frames*** (indicated using ***primes*** in the figures below) are introduced. Note when all angles are ***zero***, the body-fixed unit vector set $(\underline{n}_1, \underline{n}_2, \underline{n}_3)$ is ***aligned*** with base-fixed set $(\underline{N}_1, \underline{N}_2, \underline{N}_3)$, and all rotations are assumed to be right-handed (or dextral) rotations.
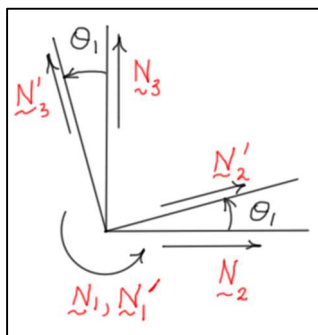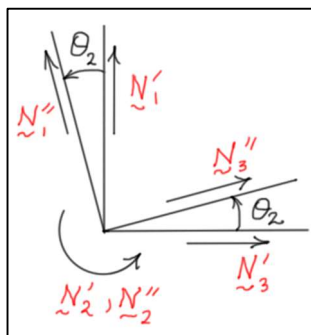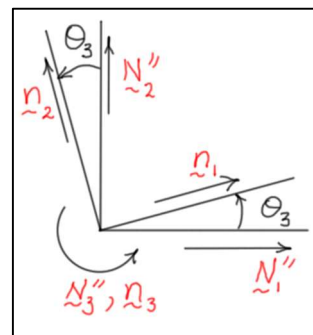
First rotation, θ_1 Second rotation, θ_2 Third rotation, θ_3

The matrix equations associated with these three successive rotations are

$$
\begin{Bmatrix} N_1' \\ N_2' \\ N_3' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & S_1 \\ 0 & -S_1 & C_1 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{bmatrix} R_1 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}
\qquad
\begin{Bmatrix} N_1'' \\ N_2'' \\ N_3'' \end{Bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix} \begin{Bmatrix} N_1' \\ N_2' \\ N_3' \end{Bmatrix} = \begin{bmatrix} R_2 \end{bmatrix} \begin{Bmatrix} N_1' \\ N_2' \\ N_3' \end{Bmatrix}
$$

$$
\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} N_1'' \\ N_2'' \\ N_3'' \end{Bmatrix} = \begin{bmatrix} R_3 \end{bmatrix} \begin{Bmatrix} N_1'' \\ N_2'' \\ N_3'' \end{Bmatrix}
$$

Here, the symbols S_i and C_i $(i = 1,2,3)$ represent the sines and cosines of the angles θ_i $(i = 1,2,3)$. These equations can be **combined** into a single matrix relationship between the **base-fixed** and the **body-fixed** unit vectors.

$$
\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{bmatrix} R_3 \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}
\tag{1}
$$

So, for a body-fixed 1-2-3 rotation sequence, the **transformation matrix** that relates the unit vectors in the body reference frame to those in the base reference frame is

$$
\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} R_3 \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & S_1 \\ 0 & -S_1 & C_1 \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} C_2C_3 & C_1S_3 + S_1S_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix}
\tag{2}
$$

As with the individual rotation matrices $\begin{bmatrix} R_i \end{bmatrix}$ $(i = 1,2,3)$, the combined matrix $\begin{bmatrix} R \end{bmatrix}$ of Eq. (2) is an **orthogonal matrix** whose determinant is +1. So, again it is easy to **invert** the relationship between the unit vector sets.

Relationship Between Body- and Base-Fixed Vector Components

To find the relationship between **vector components** in two different systems, consider the vector $\underset{\sim}{V}$ shown in the figure. If $\underset{\sim}{V}$ is **most conveniently expressed** in terms of unit vectors in the **body frame**, that is, $\underset{\sim}{V} = v_1 \underset{\sim}{n}_1 + v_2 \underset{\sim}{n}_2 + v_3 \underset{\sim}{n}_3$, then using matrix notation,

$$
\underset{\sim}{V} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}
$$
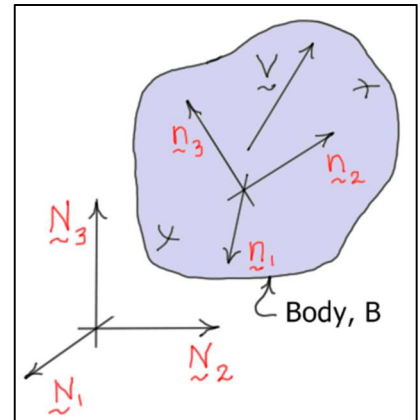
Comparing the matrices multiplying the base unit vectors gives

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} [R] = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$

**Right multiplying** both sides by the transpose of $[R]$, and then taking the **transpose** of both sides gives

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [R] \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} \tag{3}$$

Clearly, the **vector components** are transformed in the same way as the unit vectors. Matrix $[R]$ converts base-frame components into body-frame components. Because $[R]^T$ is the inverse of $[R]$, it is used to convert body-frame components into base-frame components. As the analyst is most often working with vector components in a computational setting, this is a very useful relationship.

Transformation Matrices and Direction Cosines

The **elements** of a **transformation matrix** that relates the unit vectors of two different reference frames are the **direction cosines** associated with the various unit vector pairs. Using r_{ij} to represent the **elements** of the transformation matrix $[R]$, it can be shown that

$$r_{ij} = \underset{\sim}{n}_i \cdot \underset{\sim}{N}_j = \cos\left(\theta_{ij}\right)$$
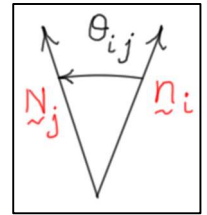
Here θ_{ij} represents the **angle between** unit vectors $\underset{\sim}{n}_i$ and $\underset{\sim}{N}_j$. Consequently, the transformation matrix $[R]$ is also referred to as the **matrix** of **direction cosines**.

Conversion of Direction Cosines to 1-2-3 Body-Fixed Angle Sequence

Given the elements of the coordinate transformation matrix $[R]$ (i.e. the direction cosines), the associated **orientation angles** can be computed. For example, using Eq. (2) for a 1-2-3 angle sequence, $[R]$ can be written as follows.

$$[R] = \begin{bmatrix} C_2C_3 & C_1S_3 + S_1S_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix} \tag{4}$$

The three orientation angles can then be calculated from the elements of the matrix by observing that

$$\theta_1 = \tan^{-1}\left(\frac{-R_{32}}{R_{33}}\right) \qquad \theta_2 = \sin^{-1}(R_{31}) \qquad \theta_3 = \tan^{-1}\left(\frac{-R_{21}}{R_{11}}\right) \tag{5}$$

Note that two of Eqs. (5) are singular when $\cos(\theta_2) = 0$, that is, when $\theta_2 = \frac{\pi}{2}$. In this case,

$$R_{12} = C_1 S_3 + S_1 C_3 = \sin(\theta_1 + \theta_3) \tag{6}$$

$$R_{22} = C_1 C_3 - S_1 S_3 = \cos(\theta_1 + \theta_3) \tag{7}$$

So, when $\theta_2 = \frac{\pi}{2}$, Eqs. (6) and (7) can be used to **solve for the sum** of the other two angles as follows.

$$\theta_1 + \theta_3 = \tan^{-1}\left(\frac{R_{12}}{R_{22}}\right) \tag{8}$$

All orientation angle sequences exhibit a ***singularity*** at some position. A procedure like that presented in Eqs. (6) through (8) can be used for those cases as well.

Angular Velocity and Orientation Angles

When using angle sequences to describe the orientation of a rigid body, the ***summation rule*** for angular velocities is used to find the angular velocity of the body. Consider the case where the orientation is given by a body-fixed, 1-2-3 rotation sequence. In that case, the angular velocity of the body can be written as

$$^R\omega_B = {}^R\omega_{R'} + {}^{R'}\omega_{R''} + {}^{R''}\omega_B = \dot{\theta}_1 N_1 + \dot{\theta}_2 N_2' + \dot{\theta}_3 N_3'' = \dot{\theta}_1 N_1' + \dot{\theta}_2 N_2'' + \dot{\theta}_3 n_3$$

Recall that the "***primed***" axes are ***intermediate*** axes introduced so the angular velocity components are all "***simple***". To make the form of $^R\omega_B$ most useful, it should be expressed in either the base reference frame $R : (N_1, N_2, N_3)$ or in the body reference frame $B : (n_1, n_2, n_3)$. For example, the ***body-fixed components*** can be found as follows

$$^R\omega_B = {}^R\omega_{R'} + {}^{R'}\omega_{R''} + {}^{R''}\omega_B = \dot{\theta}_1 N_1' + \dot{\theta}_2 N_2'' + \dot{\theta}_3 n_3 = \dot{\theta}_1 \underbrace{\left(C_2 N_1'' + S_2 N_3''\right)}_{N_1'} + \dot{\theta}_2 \underbrace{\left(S_3 n_1 + C_3 n_2\right)}_{N_2''} + \dot{\theta}_3 n_3$$

$$= \dot{\theta}_1 C_2 \underbrace{\left(C_3 n_1 - S_3 n_2\right)}_{N_1''} + \left(\dot{\theta}_1 S_2\right) n_3 + \dot{\theta}_2 \left(S_3 n_1 + C_3 n_2\right) + \dot{\theta}_3 n_3$$

$$= \left(\dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3\right) n_1 + \left(-\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3\right) n_2 + \left(\dot{\theta}_1 S_2 + \dot{\theta}_3\right) n_3$$

$$\triangleq \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3$$

So, the body-fixed angular velocity components for a 1-2-3 body-fixed rotation sequence are

$$\begin{array}{l} \omega_1 = \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3 \\ \omega_2 = -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \\ \omega_3 = \dot{\theta}_1 S_2 + \dot{\theta}_3 \end{array} \tag{9}$$

These equations can be ***inverted*** to solve for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of ω_1, ω_2, and ω_3 to give

$$\dot{\theta}_1 = \left(\omega_1 C_3 - \omega_2 S_3\right)/C_2$$
$$\dot{\theta}_2 = \omega_1 S_3 + \omega_2 C_3$$
$$\dot{\theta}_3 = \omega_3 - S_2\left(\omega_1 C_3 - \omega_2 S_3\right)/C_2$$

(10)

Eqs. (9) and (10) are the ***kinematic equations*** for ***angular motion*** of a rigid body. A similar process can be followed to find the ***base-fixed*** components of $^R\underset{\sim}{\omega}_B$.

Note the first and third of Eqs. (10) shown above are ***singular*** when $C_2 \triangleq \cos\left(\theta_2\right)$ is ***zero***, that is, when the ***second orientation angle*** is 90 degrees. As mentioned previously, all ***orientation angle sequences*** display such a ***singularity*** at some position. This can cause problems for computer programs that use angle sequences to describe the orientation of rigid bodies. Orientation parameters discussed in Unit 6 of this volume can be used to remedy this situation.

Angular Acceleration Revisited

As presented in Unit 1, the angular acceleration of a body is found by ***direct differentiation*** of the angular velocity vector. In the presentation above, the angular velocity is expressed using ***body-fixed*** unit vectors, and the angular acceleration can be found using the "***derivative rule***" as follows.

$$^R\underset{\sim}{\alpha}_B = \frac{^R d}{dt}\left(^R\underset{\sim}{\omega}_B\right) = \frac{^B d}{dt}\left(^R\underset{\sim}{\omega}_B\right) + \underbrace{\left(^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{\omega}_B\right)}_{\text{zero}} = \frac{^B d}{dt}\left(^R\underset{\sim}{\omega}_B\right)$$

(11)

The angular velocity vector is ***unique*** in this way. It is the ***only*** vector whose derivative in the base frame is the same as its derivative in the body frame.

Example 1

The system shown has three components, a vertical column C, a horizontal arm M, and a disk D. An arbitrary orientation of D can be described by the three angles θ_1, θ_2, and θ_3 as shown in the diagram. Four reference frames are used to define the orientation of D in R: the ground frame $R:\left(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}\right)$, the column frame $C:\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$, the arm frame $M:\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$, and the disk frame $D:\left(\underset{\sim}{d}_1, \underset{\sim}{d}_2, \underset{\sim}{d}_3\right)$. When $\theta_i\ (i=1,2,3)$ are zero, all the frames are aligned so that $\underset{\sim}{i}=\underset{\sim}{e}_1=\underset{\sim}{n}_1=\underset{\sim}{d}_1$, $\underset{\sim}{j}=\underset{\sim}{e}_2=\underset{\sim}{n}_2=\underset{\sim}{d}_2$, and $\underset{\sim}{k}=\underset{\sim}{e}_3=\underset{\sim}{n}_3=\underset{\sim}{d}_3$. Then, a non-zero angle θ_1 is a "2" rotation that
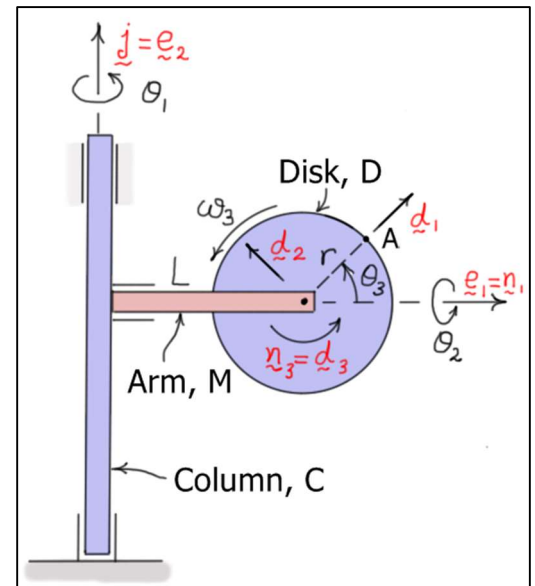
transforms R into C, angle θ_2 is a "1" rotation that transforms C into M, and angle θ_3 is a "3" rotation that transforms M into D.

Find: (express vector results in frame D)

a) $[R]$ the transformation matrix that converts ground frame components into disk frame components

b) $^R\underset{\sim}{\omega}_D$ and $^R\underset{\sim}{\alpha}_D$ the angular velocity and the angular acceleration of D in R

Solution:

a) θ_1 rotation:
$$\begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = \begin{bmatrix} C_1 & 0 & -S_1 \\ 0 & 1 & 0 \\ S_1 & 0 & C_1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix} = [R_1] \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix} \quad\Rightarrow\quad \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix} = [R_1]^T \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}$$

θ_2 rotation:
$$\begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2 & S_2 \\ 0 & -S_2 & C_2 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = [R_2] \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} \quad\Rightarrow\quad \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = [R_2]^T \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix}$$

θ_3 rotation:
$$\begin{Bmatrix} \underset{\sim}{d}_1 \\ \underset{\sim}{d}_2 \\ \underset{\sim}{d}_3 \end{Bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = [R_3] \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} \quad\Rightarrow\quad \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = [R_3]^T \begin{Bmatrix} \underset{\sim}{d}_1 \\ \underset{\sim}{d}_2 \\ \underset{\sim}{d}_3 \end{Bmatrix}$$

So, the base unit vectors can be related directly to those fixed in D as follows

$$\begin{Bmatrix} \underset{\sim}{d}_1 \\ \underset{\sim}{d}_2 \\ \underset{\sim}{d}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix}$$

$$[R] = \underbrace{\begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[R_3]} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2 & S_2 \\ 0 & -S_2 & C_2 \end{bmatrix}}_{[R_2]} \underbrace{\begin{bmatrix} C_1 & 0 & -S_1 \\ 0 & 1 & 0 \\ S_1 & 0 & C_1 \end{bmatrix}}_{[R_1]} = \begin{bmatrix} C_3 & S_3 C_2 & S_3 S_2 \\ -S_3 & C_3 C_2 & C_3 S_2 \\ 0 & -S_2 & C_2 \end{bmatrix} \begin{bmatrix} C_1 & 0 & -S_1 \\ 0 & 1 & 0 \\ S_1 & 0 & C_1 \end{bmatrix}$$

$$\Rightarrow \quad \boxed{[R] = \begin{bmatrix} S_1 S_2 S_3 + C_1 C_3 & C_2 S_3 & C_1 S_2 S_3 - S_1 C_3 \\ S_1 S_2 C_3 - C_1 S_3 & C_2 C_3 & C_1 S_2 C_3 + S_1 S_3 \\ S_1 C_2 & -S_2 & C_1 C_2 \end{bmatrix}} \quad \text{(2-1-3 body-fixed rotation sequence)}$$

b) Using the summation rule for angular velocities

$$^R\underset{\sim}{\omega}_D = {}^R\underset{\sim}{\omega}_F + {}^F\underset{\sim}{\omega}_M + {}^M\underset{\sim}{\omega}_D = \dot{\theta}_1 \underset{\sim}{e}_2 + \dot{\theta}_2 \underset{\sim}{n}_1 + \dot{\theta}_3 \underset{\sim}{d}_3$$

$$= \dot{\theta}_1 \left(C_2 \underset{\sim}{n}_2 - S_2 \underset{\sim}{n}_3 \right) + \dot{\theta}_2 \left(C_3 \underset{\sim}{d}_1 - S_3 \underset{\sim}{d}_2 \right) + \dot{\theta}_3 \underset{\sim}{d}_3$$

$$= \dot{\theta}_1 C_2 \left(S_3 \underset{\sim}{d}_1 + C_3 \underset{\sim}{d}_2 \right) - \dot{\theta}_1 S_2 \underset{\sim}{d}_3 + \dot{\theta}_2 \left(C_3 \underset{\sim}{d}_1 - S_3 \underset{\sim}{d}_2 \right) + \dot{\theta}_3 \underset{\sim}{d}_3$$

$$\Rightarrow \quad {}^R\underset{\sim}{\omega}_D = \left(\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3\right)\underset{\sim}{d}_1 + \left(\dot{\theta}_1 C_2 C_3 - \dot{\theta}_2 S_3\right)\underset{\sim}{d}_2 + \left(\dot{\theta}_3 - \dot{\theta}_1 S_2\right)\underset{\sim}{d}_3$$

So, the **body-fixed angular velocity components** for a 2-1-3 body-fixed rotation sequence are

$$\begin{aligned}
\omega_1 &= \dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \\
\omega_2 &= \dot{\theta}_1 C_2 C_3 - \dot{\theta}_2 S_3 \\
\omega_3 &= \dot{\theta}_3 - \dot{\theta}_1 S_2
\end{aligned}$$ (2-1-3 body-fixed rotation sequence)

Differentiating to find the angular acceleration gives

$$ {}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \frac{{}^B d}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \dot{\omega}_1 \underset{\sim}{d}_1 + \dot{\omega}_2 \underset{\sim}{d}_2 + \dot{\omega}_3 \underset{\sim}{d}_3 $$

Here,

$$\begin{aligned}
\dot{\omega}_1 &= \ddot{\theta}_1 C_2 S_3 - \dot{\theta}_1 \dot{\theta}_2 S_2 S_3 + \dot{\theta}_1 \dot{\theta}_3 C_2 C_3 + \ddot{\theta}_2 C_3 - \dot{\theta}_2 \dot{\theta}_3 S_3 \\
\dot{\omega}_2 &= \ddot{\theta}_1 C_2 C_3 - \dot{\theta}_1 \dot{\theta}_2 S_2 C_3 - \dot{\theta}_1 \dot{\theta}_3 C_2 S_3 - \ddot{\theta}_2 S_3 - \dot{\theta}_2 \dot{\theta}_3 C_3 \\
\dot{\omega}_3 &= \ddot{\theta}_3 - \ddot{\theta}_1 S_2 - \dot{\theta}_1 \dot{\theta}_2 C_2
\end{aligned}$$

Example 2

The orientation of an aircraft can be defined by a 3-2-1 body-fixed rotation sequence. As before, the body axes (x_b, y_b, z_b) are initially aligned with the fixed frame axes (X, Y, Z). It is common to refer to these the angles as ψ, θ, and ϕ. For small angles these are equivalent to the "yaw", "pitch", and "roll" angles of the aircraft.

Find: (express vector results in frame B)
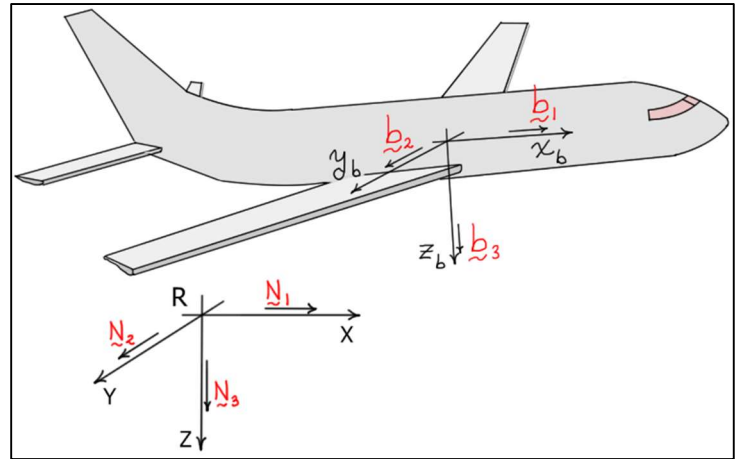
a) $[R]$, the transformation matrix that converts ground frame components into body frame components

b) ${}^R\underset{\sim}{\omega}_B$ and ${}^R\underset{\sim}{\alpha}_B$ the angular velocity and the angular acceleration of B in R

Solution:

a) ψ rotation:
$$\begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R_1] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R_1]^T \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix}$$

θ rotation:
$$\begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = [R_2] \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \underset{\sim}{N}_1' \\ \underset{\sim}{N}_2' \\ \underset{\sim}{N}_3' \end{Bmatrix} = [R_2]^T \begin{Bmatrix} \underset{\sim}{N}_1'' \\ \underset{\sim}{N}_2'' \\ \underset{\sim}{N}_3'' \end{Bmatrix}$$

ϕ rotation:
$$\begin{Bmatrix} \underset{\sim}{b_1} \\ \underset{\sim}{b_2} \\ \underset{\sim}{b_3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N_1''} \\ \underset{\sim}{N_2''} \\ \underset{\sim}{N_3''} \end{Bmatrix} = [R_3] \begin{Bmatrix} \underset{\sim}{N_1''} \\ \underset{\sim}{N_2''} \\ \underset{\sim}{N_3''} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \underset{\sim}{N_1''} \\ \underset{\sim}{N_2''} \\ \underset{\sim}{N_3''} \end{Bmatrix} = [R_3]^T \begin{Bmatrix} \underset{\sim}{b_1} \\ \underset{\sim}{b_2} \\ \underset{\sim}{b_3} \end{Bmatrix}$$

So,

$$\begin{Bmatrix} \underset{\sim}{b_1} \\ \underset{\sim}{b_2} \\ \underset{\sim}{b_3} \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} \underset{\sim}{N_1} \\ \underset{\sim}{N_2} \\ \underset{\sim}{N_3} \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{N_1} \\ \underset{\sim}{N_2} \\ \underset{\sim}{N_3} \end{Bmatrix}$$

$$[R] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{bmatrix}}_{[R_3]} \underbrace{\begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}}_{[R_2]} \underbrace{\begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[R_1]} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ S_\theta S_\phi & C_\phi & C_\theta S_\phi \\ S_\theta C_\phi & -S_\phi & C_\theta C_\phi \end{bmatrix} \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [R] = \begin{bmatrix} C_\psi C_\theta & S_\psi C_\theta & -S_\theta \\ C_\psi S_\theta S_\phi - S_\psi C_\phi & S_\psi S_\theta S_\phi + C_\psi C_\phi & C_\theta S_\phi \\ C_\psi S_\theta C_\phi + S_\psi S_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi & C_\theta C_\phi \end{bmatrix} \quad \text{(3-2-1 body-fixed rotation sequence)}$$

b) Using the summation rule for angular velocities

$${}^R\underset{\sim}{\omega}_B = {}^R\underset{\sim}{\omega}_{R'} + {}^{R'}\underset{\sim}{\omega}_{R''} + {}^{R''}\underset{\sim}{\omega}_B = \dot\psi \underset{\sim}{N_3'} + \dot\theta \underset{\sim}{N_2''} + \dot\phi \underset{\sim}{b_1} = \dot\psi \underbrace{\left(-S_\theta \underset{\sim}{N_1''} + C_\theta \underset{\sim}{N_3''} \right)}_{\underset{\sim}{N_3'}} + \dot\theta \underbrace{\left(C_\phi \underset{\sim}{b_2} - S_\phi \underset{\sim}{b_3} \right)}_{\underset{\sim}{N_2''}} + \dot\phi \underset{\sim}{b_1}$$

$$= -\left(\dot\psi S_\theta \right) \underset{\sim}{b_1} + \dot\psi C_\theta \underbrace{\left(S_\phi \underset{\sim}{b_2} + C_\phi \underset{\sim}{b_3} \right)}_{\underset{\sim}{N_3''}} + \dot\theta \left(C_\phi \underset{\sim}{b_2} - S_\phi \underset{\sim}{b_3} \right) + \dot\phi \underset{\sim}{b_1}$$

$$= \left(\dot\phi - \dot\psi S_\theta \right) \underset{\sim}{b_1} + \left(\dot\theta C_\phi + \dot\psi C_\theta S_\phi \right) \underset{\sim}{b_2} + \left(-\dot\theta S_\phi + \dot\psi C_\theta C_\phi \right) \underset{\sim}{b_3}$$

$$\Rightarrow \quad {}^R\underset{\sim}{\omega}_B = \left(\dot\phi - \dot\psi S_\theta \right) \underset{\sim}{b_1} + \left(\dot\theta C_\phi + \dot\psi C_\theta S_\phi \right) \underset{\sim}{b_2} + \left(-\dot\theta S_\phi + \dot\psi C_\theta C_\phi \right) \underset{\sim}{b_3}$$

So, the **body-fixed angular velocity components** for a 3-2-1 body-fixed rotation sequence are

$$\begin{aligned} \omega_1 &= \dot\phi - \dot\psi S_\theta \\ \omega_2 &= \dot\theta C_\phi + \dot\psi C_\theta S_\phi \\ \omega_3 &= -\dot\theta S_\phi + \dot\psi C_\theta C_\phi \end{aligned} \quad \text{(3-2-1 body-fixed rotation sequence)}$$

Differentiating to find the angular acceleration

$$\boxed{{}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B \right) = \frac{{}^B d}{dt}\left({}^R\underset{\sim}{\omega}_B \right) = \dot\omega_1 \underset{\sim}{b_1} + \dot\omega_2 \underset{\sim}{b_2} + \dot\omega_3 \underset{\sim}{b_3}}$$

Here,

$$\dot{\omega}_1 = \ddot{\phi} - \ddot{\psi}\, S_\theta - \dot{\psi}\, \dot{\theta}\, C_\theta$$

$$\dot{\omega}_2 = \ddot{\theta}\, C_\phi - \dot{\theta}\, \dot{\phi}\, S_\phi + \ddot{\psi}\, C_\theta\, S_\phi - \dot{\psi}\, \dot{\theta}\, S_\theta\, S_\phi + \dot{\psi}\, \dot{\phi}\, C_\theta\, C_\phi$$

$$\dot{\omega}_3 = -\ddot{\theta}\, S_\phi - \dot{\theta}\, \dot{\phi}\, C_\phi + \ddot{\psi}\, C_\theta\, C_\phi - \dot{\psi}\, \dot{\theta}\, S_\theta\, C_\phi - \dot{\psi}\, \dot{\phi}\, C_\theta\, S_\phi$$

Notes

1. The subscript on the angle represents the **order** of the rotation sequence **not** the **axis** about which the rotation occurs. So, for example, in a 3-1-2 rotation sequence, θ_1 refers to a "3" rotation, θ_2 refers to a "1" rotation, and θ_3 refers to a "2" rotation. The angle sequence is always $\theta_1 \rightarrow \theta_2 \rightarrow \theta_3$, but the axes about which the rotations occur differ from sequence to sequence.

2. Transformation matrices for many different **body-fixed** rotation sequences are given in Appendix I of the text *Spacecraft Dynamics* by T. R. Kane, P. W. Likins, and D. A. Levinson, McGraw-Hill, 1983. Specifically, results are given for 1-2-3, 2-3-1, 3-1-2, 1-3-2, 2-1-3, 3-2-1, 1-2-1, 1-3-1, 2-1-2, 2-3-2, 3-1-3, and 3-2-3 rotation sequences. The transformation matrices reported here are the transposes of those presented in *Spacecraft Dynamics*.

3. Appendix I of *Spacecraft Dynamics* also gives transformation matrices for many different **base-fixed** rotation sequences as well. This text will primarily use body-fixed rotation sequences.

4. Appendix II of *Spacecraft Dynamics* gives relationships between the **body-fixed angular velocity components** and the **time derivatives of the orientation angles** for all the body-fixed and base-fixed rotation sequences as well. The body-fixed components of the angular velocity vector are useful in the calculation of angular momentum of a rigid body.

Exercises

5.1 The system shown consists of three components, the arms A and B and the end-effector E. The orientation of E relative to a fixed frame is described by the three angles shown. Note that the sequence of rotations $(\theta_1, \theta_2, \theta_3)$ is a 2-3-1 body-fixed rotation sequence. a) Derive the transformation matrix $[R]$ that relates the unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ (fixed in E) to the unit vectors $(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$ of the fixed frame. b) Find the $\underset{\sim}{e}_i$ $(i=1,2,3)$
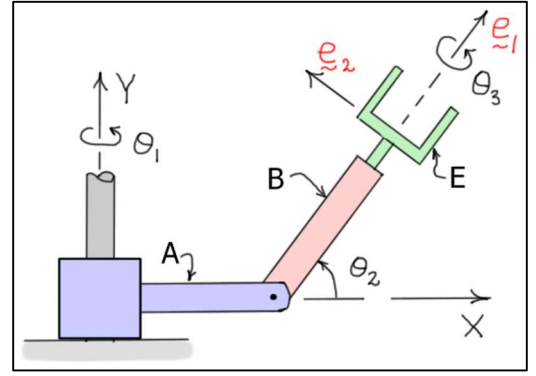
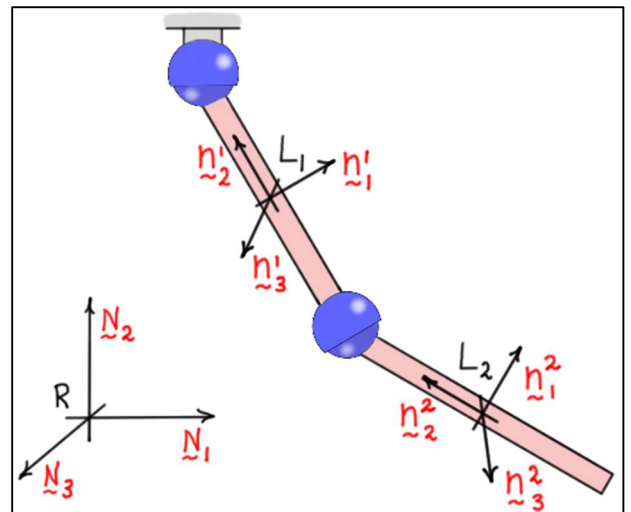components of ${}^R\underset{\sim}{\omega}_E$ the angular velocity of E in R. c) Invert the equations from part (b) to solve for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of the angular velocity components. d) Find ${}^R\underset{\sim}{\alpha}_E$ the angular acceleration of E relative to the fixed frame.

Answers:
$$[R] = \begin{bmatrix} C_1 C_2 & S_2 & -S_1 C_2 \\ S_1 S_3 - C_1 S_2 C_3 & C_2 C_3 & C_1 S_3 + S_1 S_2 C_3 \\ S_1 C_3 + C_1 S_2 S_3 & -C_2 S_3 & C_1 C_3 - S_1 S_2 S_3 \end{bmatrix}$$

$$\begin{aligned} \omega_1 &= \dot{\theta}_3 + \dot{\theta}_1 S_2 \\ \omega_2 &= \dot{\theta}_2 S_3 + \dot{\theta}_1 C_2 C_3 \\ \omega_3 &= \dot{\theta}_2 C_3 - \dot{\theta}_1 C_2 S_3 \end{aligned} \Rightarrow \begin{aligned} \dot{\theta}_1 &= (\omega_2 C_3 - \omega_3 S_3)/C_2 \\ \dot{\theta}_2 &= \omega_2 S_3 + \omega_3 C_3 \\ \dot{\theta}_3 &= \omega_1 + S_2(\omega_3 S_3 - \omega_2 C_3)/C_2 \end{aligned}$$

$$\begin{aligned} \dot{\omega}_1 &= \ddot{\theta}_3 + \ddot{\theta}_1 S_2 + \dot{\theta}_1 \dot{\theta}_2 C_2 \\ \dot{\omega}_2 &= \ddot{\theta}_2 S_3 + \dot{\theta}_2 \dot{\theta}_3 C_3 + \ddot{\theta}_1 C_2 C_3 - \dot{\theta}_1 \dot{\theta}_2 S_2 C_3 - \dot{\theta}_1 \dot{\theta}_3 C_2 S_3 \\ \dot{\omega}_3 &= \ddot{\theta}_2 C_3 - \dot{\theta}_2 \dot{\theta}_3 S_3 - \ddot{\theta}_1 C_2 S_3 + \dot{\theta}_1 \dot{\theta}_2 S_2 S_3 - \dot{\theta}_1 \dot{\theta}_3 C_2 C_3 \end{aligned}$$

5.2 The system shown is a three-dimensional double pendulum. The first link is connected to ground and the second link is connected to the first with ball and socket joints. The ground frame is $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$ and the link frames are $L_i:(\underset{\sim}{n}_1^i, \underset{\sim}{n}_2^i, \underset{\sim}{n}_3^i)$ $(i=1,2)$. The orientation of each link is defined relative to R using a 3-1-3 body-fixed rotation sequence. a) Derive the transformation matrices $[R_i]$ $(i=1,2)$ that relate the unit vectors $(\underset{\sim}{n}_1^i, \underset{\sim}{n}_2^i, \underset{\sim}{n}_3^i)$ $(i=1,2)$ fixed in the two links to the unit

vectors $\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ fixed in the ground. b) Find the body-fixed components of $^R\underset{\sim}{\omega}_{L_i}$ the angular velocities of the links relative to R. c) Invert the equations from part (b) to solve for $\dot{\theta}_{i1}$, $\dot{\theta}_{i2}$, and $\dot{\theta}_{i3}$ in terms of the angular velocity components. d) Find $^R\underset{\sim}{\alpha}_{L_i}$ the angular accelerations of the links relative to R.

Answers:
$$\left[R_i \right] = \begin{bmatrix} C_{i1}C_{i3} - S_{i1}C_{i2}S_{i3} & S_{i1}C_{i3} + C_{i1}C_{i2}S_{i3} & S_{i2}S_{i3} \\ -C_{i1}S_{i3} - S_{i1}C_{i2}C_{i3} & -S_{i1}S_{i3} + C_{i1}C_{i2}C_{i3} & S_{i2}C_{i3} \\ S_{i1}S_{i2} & -C_{i1}S_{i2} & C_{i2} \end{bmatrix}$$

$$\begin{aligned} \omega_{i1} &= \dot{\theta}_{i1}S_{i2}S_{i3} + \dot{\theta}_{i2}C_{i3} \\ \omega_{i2} &= \dot{\theta}_{i1}S_{i2}C_{i3} - \dot{\theta}_{i2}S_{i3} \\ \omega_{i3} &= \dot{\theta}_{i3} + \dot{\theta}_{i1}C_{i2} \end{aligned} \Rightarrow \begin{aligned} \dot{\theta}_{i1} &= \left(\omega_{i1}S_{i3} + \omega_{i2}C_{i3}\right)/S_{i2} \\ \dot{\theta}_{i2} &= \omega_{i1}C_{i3} - \omega_{i2}S_{i3} \\ \dot{\theta}_{i3} &= \omega_{i3} - C_{i2}\left(\omega_{i1}S_{i3} + \omega_{i2}C_{i3}\right)/S_{i2} \end{aligned}$$

$$\begin{aligned} \dot{\omega}_{i1} &= \ddot{\theta}_{i1}S_{i2}S_{i3} + \dot{\theta}_{i1}\dot{\theta}_{i2}C_{i2}S_{i3} + \dot{\theta}_{i1}\dot{\theta}_{i3}S_{i2}C_{i3} + \ddot{\theta}_{i2}C_{i3} - \dot{\theta}_{i2}\dot{\theta}_{i3}S_{i3} \\ \dot{\omega}_{i2} &= \ddot{\theta}_{i1}S_{i2}C_{i3} + \dot{\theta}_{i1}\dot{\theta}_{i2}C_{i2}C_{i3} - \dot{\theta}_{i1}\dot{\theta}_{i3}S_{i2}S_{i3} - \ddot{\theta}_{i2}S_{i3} - \dot{\theta}_{i2}\dot{\theta}_{i3}C_{i3} \\ \dot{\omega}_{i3} &= \ddot{\theta}_{i3} + \ddot{\theta}_{i1}C_{i2} - \dot{\theta}_{i1}\dot{\theta}_{i2}S_{i2} \end{aligned}$$

References:

1. T.R. Kane, P.W. Likins, and D.A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983
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