An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume III: Introduction to Multibody Kinematics

Unit 2

Angular Velocity and Partial Angular Velocity

Summary

This unit focuses on the ***matrix-based*** calculation of ***vector*** components of ***angular velocity*** and ***partial angular velocity matrices***. The calculations are performed using ***fixed frame*** and ***body frame*** components and are based on ***absolute*** and ***relative coordinates***. Both ***orientation angle derivatives*** and ***angular velocity*** components are used as ***generalized speeds***. Algorithms are developed for the efficient calculation of these quantities for multibody systems.

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Introduction

As presented in Unit 1 of this volume, the ***degrees of freedom*** of a multibody system can be represented by ***absolute*** coordinates, ***relative*** coordinates, or both. As defined herein, ***absolute coordinates*** are measured relative to a ***fixed frame***, and ***relative coordinates*** are measured relative to ***other bodies*** in the system. This unit focuses on ***matrix-based*** calculations of ***angular velocities*** and ***partial angular velocities*** in terms of both ***absolute*** and ***relative*** coordinates. The vector components are resolved in both ***fixed frames*** and ***body*** (rotating) ***frames***. Both ***angle derivatives*** and ***angular velocity*** components are considered as ***generalized speeds***.

Angular Velocity & Partial Angular Velocity Using Absolute Coordinates

Angular Velocity Using a 1-2-3 Body Fixed Rotation Sequence $\left(\theta_{B1},\theta_{B2},\theta_{B3}\right)$

To describe the ***orientation*** of rigid body $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ of a multibody system ***relative*** to a ***fixed reference frame*** $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ using a ***body fixed orientation angle sequence***, a set of intermediate reference frames $R':\left(\underset{\sim}{N}'_1,\underset{\sim}{N}'_2,\underset{\sim}{N}'_3\right)$ and $R'':\left(\underset{\sim}{N}''_1,\underset{\sim}{N}''_2,\underset{\sim}{N}''_3\right)$ can be defined as shown in Unit 5 of Volume I. These intermediate reference frames can be used to calculate the angular velocities of bodies. For example, using a 1-2-3 body fixed rotation sequence, the angular velocity of body B can be written as follows.

$$\boxed{{}^{R}\underset{\sim}{\omega}_B = {}^{R}\underset{\sim}{\omega}_{R'} + {}^{R'}\underset{\sim}{\omega}_{R''} + {}^{R''}\underset{\sim}{\omega}_B = \dot{\theta}_{B1}\underset{\sim}{N}_1 + \dot{\theta}_{B2}\underset{\sim}{N}'_2 + \dot{\theta}_{B3}\underset{\sim}{N}''_3} \qquad (1)$$

If $\left[{}^{R}R_{R'}\right]$ is the transformation matrix that describes the orientation of frame R' relative to frame R and $\left[{}^{R'}R_{R''}\right]$ is the transformation matrix that describes the orientation of frame R'' relative to frame R', then the following equations can be written relating the unit vectors in each of the frames.

$$\boxed{\begin{Bmatrix}\underset{\sim}{N}'_1\\ \underset{\sim}{N}'_2\\ \underset{\sim}{N}'_3\end{Bmatrix} = \left[{}^{R}R_{R'}\right]\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix}} \quad \text{and} \quad \boxed{\begin{Bmatrix}\underset{\sim}{N}''_1\\ \underset{\sim}{N}''_2\\ \underset{\sim}{N}''_3\end{Bmatrix} = \left[{}^{R'}R_{R''}\right]\begin{Bmatrix}\underset{\sim}{N}'_1\\ \underset{\sim}{N}'_2\\ \underset{\sim}{N}'_3\end{Bmatrix} = \left[{}^{R'}R_{R''}\right]\left[{}^{R}R_{R'}\right]\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix}}$$

Here, for a ***1-2-3 body fixed rotation sequence***

$$\boxed{\left[{}^{R}R_{R'}\right] = \begin{bmatrix}1 & 0 & 0\\ 0 & C_{B1} & S_{B1}\\ 0 & -S_{B1} & C_{B1}\end{bmatrix}} \quad \boxed{\left[{}^{R'}R_{R''}\right] = \begin{bmatrix}C_{B2} & 0 & -S_{B2}\\ 0 & 1 & 0\\ S_{B2} & 0 & C_{B2}\end{bmatrix}}$$

$$\boxed{\left[{}^{R'}R_{R''}\right]\left[{}^{R}R_{R'}\right] = \begin{bmatrix}C_{B2} & 0 & -S_{B2}\\ 0 & 1 & 0\\ S_{B2} & 0 & C_{B2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & C_{B1} & S_{B1}\\ 0 & -S_{B1} & C_{B1}\end{bmatrix} = \begin{bmatrix}C_{B2} & S_{B1}S_{B2} & -C_{B1}S_{B2}\\ 0 & C_{B1} & S_{B1}\\ S_{B2} & -S_{B1}C_{B2} & C_{B1}C_{B2}\end{bmatrix}}$$

Here, S_{Bi} and C_{Bi} $(i=1,2,3)$ represent the sines and cosines of the angles $\theta_{Bi}\,(i=1,2,3)$.

The ***fixed frame*** components of $\underset{\sim}{N}'_2$ are given by the ***second row*** of $\left[{}^{R}R_{R'} \right]$, and the ***fixed frame*** components $\underset{\sim}{N}''_3$ are given by the ***third row*** of $\left[{}^{R'}R_{R''} \right]\left[{}^{R}R_{R'} \right]$. Hence, the ***fixed frame*** components of the ***angular velocity vector*** of body B can be written in matrix form as follows.

$$\begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} = \dot{\theta}_{B1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + \dot{\theta}_{B2} \begin{Bmatrix} 0 \\ C_{B1} \\ S_{B1} \end{Bmatrix} + \dot{\theta}_{B3} \begin{Bmatrix} S_{B2} \\ -S_{B1}C_{B2} \\ C_{B1}C_{B2} \end{Bmatrix}$$

Or,

$$\boxed{\begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & S_{B2} \\ 0 & C_{B1} & -S_{B1}C_{B2} \\ 0 & S_{B1} & C_{B1}C_{B2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{B1} \\ \dot{\theta}_{B2} \\ \dot{\theta}_{B3} \end{Bmatrix}} \quad \text{(fixed frame components)} \qquad (2)$$

Note that the ***first column*** of the ***coefficient matrix*** holds the fixed frame components of $\underset{\sim}{N}_1$, the ***second column*** holds the fixed frame components of $\underset{\sim}{N}'_2$, and the ***third column*** holds the fixed frame components of $\underset{\sim}{N}''_3$.

The ***same approach*** can be used to determine an equation for ***body frame*** components. In that case, write

$$\boxed{{}^{R}\underset{\sim}{\omega}_{B} = {}^{R}\underset{\sim}{\omega}_{R'} + {}^{R'}\underset{\sim}{\omega}_{R''} + {}^{R''}\underset{\sim}{\omega}_{B} = \dot{\theta}_{B1}\underset{\sim}{N}'_1 + \dot{\theta}_{B2}\underset{\sim}{N}''_2 + \dot{\theta}_{B3}\,\underset{\sim}{e}_3}$$

If $\left[{}^{R'}R_{R''} \right]$ is the transformation matrix that describes the orientation of frame R'' relative to frame R', and $\left[{}^{R''}R_{B} \right]$ is the transformation matrix that describes the orientation of body frame relative to frame R'', then

$$\boxed{\begin{Bmatrix} \underset{\sim}{N}''_1 \\ \underset{\sim}{N}''_2 \\ \underset{\sim}{N}''_3 \end{Bmatrix} = \left[{}^{R''}R_{B} \right]^T \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}} \quad \text{and} \quad \boxed{\begin{Bmatrix} \underset{\sim}{N}'_1 \\ \underset{\sim}{N}'_2 \\ \underset{\sim}{N}'_3 \end{Bmatrix} = \left[{}^{R'}R_{R''} \right]^T \begin{Bmatrix} \underset{\sim}{N}''_1 \\ \underset{\sim}{N}''_2 \\ \underset{\sim}{N}''_3 \end{Bmatrix} = \left[{}^{R'}R_{R''} \right]^T \left[{}^{R''}R_{B} \right]^T \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}}$$

Here,

$$\boxed{\left[{}^{R''}R_{B} \right]^T = \begin{bmatrix} C_{B3} & -S_{B3} & 0 \\ S_{B3} & C_{B3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{\left[{}^{R'}R_{R''} \right]^T \left[{}^{R''}R_{B} \right]^T = \begin{bmatrix} C_{B2} & 0 & S_{B2} \\ 0 & 1 & 0 \\ -S_{B2} & 0 & C_{B2} \end{bmatrix} \begin{bmatrix} C_{B3} & -S_{B3} & 0 \\ S_{B3} & C_{B3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{B2}C_{B3} & -C_{B2}S_{B3} & S_{B2} \\ S_{B3} & C_{B3} & 0 \\ -S_{B2}C_{B3} & S_{B2}S_{B3} & C_{B2} \end{bmatrix}}$$

The ***body frame*** components of $\underset{\sim}{N}'_1$ are given by the ***first row*** of $\left[{}^{R'}R_{R''}\right]^T\left[{}^{R''}R_B\right]^T$, and the ***body frame*** components of $\underset{\sim}{N}''_2$ are given by the ***second row*** of $\left[{}^{R''}R_B\right]^T$. Hence, the ***body frame*** components of the ***angular velocity vector*** in matrix form can be written as follows.

$$\begin{Bmatrix}\omega'_{B1}\\ \omega'_{B2}\\ \omega'_{B3}\end{Bmatrix}=\dot{\theta}_{B1}\begin{Bmatrix}C_{B2}C_{B3}\\ -C_{B2}S_{B3}\\ S_{B2}\end{Bmatrix}+\dot{\theta}_{B2}\begin{Bmatrix}S_{B3}\\ C_{B3}\\ 0\end{Bmatrix}+\dot{\theta}_{B3}\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}$$

Or,

$$\begin{Bmatrix}\omega'_{B1}\\ \omega'_{B2}\\ \omega'_{B3}\end{Bmatrix}=\begin{bmatrix}C_{B2}C_{B3} & S_{B3} & 0\\ -C_{B2}S_{B3} & C_{B3} & 0\\ S_{B2} & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{B1}\\ \dot{\theta}_{B2}\\ \dot{\theta}_{B3}\end{Bmatrix}\quad \text{(body frame components)}\qquad (3)$$

Note that the ***first column*** of the coefficient matrix holds the ***body frame*** components of $\underset{\sim}{N}'_1$, the ***second column*** holds the ***body frame*** components of $\underset{\sim}{N}''_2$, and the ***third column*** holds the ***body frame*** components of $\underset{\sim}{e}_3$.

The results of Equation (3) can also be found in Appendix II of Kane, Likins, and Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983. The text has results for ***many*** other body fixed, orientation-angle sequences as well.

Partial Angular Velocities Using Orientation Angle Derivatives as Generalized Speeds

Using the ***time derivatives*** of the ***orientation angles*** as ***generalized speeds***, the ***partial angular velocities*** of body B of the multibody system are the partial derivatives of ${}^R\underset{\sim}{\omega}_B$ with respect to $\dot{\theta}_{Bi}$ $(i=1,2,3)$. Specifically,

$$\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}_{B1}}=\underset{\sim}{N}_1 \qquad \frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}_{B2}}=\underset{\sim}{N}'_2 \qquad \frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}_{B3}}=\underset{\sim}{N}''_3$$

These results can be conveniently expressed in ***fixed frame*** or ***body frame*** components. The ***fixed frame components*** of the partial angular velocity vectors can be written as follows.

$$\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}_{B1}}\to\left\{{}^R\omega_{B,\dot{\theta}_{B1}}\right\}=\begin{Bmatrix}1\\ 0\\ 0\end{Bmatrix} \qquad \frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}_{B2}}\to\left\{{}^R\omega_{B,\dot{\theta}_{B2}}\right\}=\begin{Bmatrix}0\\ C_{B1}\\ S_{B1}\end{Bmatrix} \qquad \frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot{\theta}_{B3}}\to\left\{{}^R\omega_{B,\dot{\theta}_{B3}}\right\}=\begin{Bmatrix}S_{B2}\\ -S_{B1}C_{B2}\\ C_{B1}C_{B2}\end{Bmatrix}$$

These results can be expressed in a single matrix equation as follows.

$$\left[{}^R\omega_{B,\dot{\theta}_B}\right]_{3\times3}=\begin{bmatrix}1 & 0 & S_{B2}\\ 0 & C_{B1} & -S_{B1}C_{B2}\\ 0 & S_{B1} & C_{B1}C_{B2}\end{bmatrix}\quad \text{(fixed frame components)}\qquad (4)$$

Here, $\left[{}^{R}\omega_{B,\dot{\theta}_B}\right]$ is the ***partial angular velocity matrix*** of body B with respect to the angle derivatives expressed using ***fixed frame*** components. This is the same as the coefficient matrix in Equation (2). Regarding notation, note that the notation "$,\dot{\theta}_B$" in the subscript indicates ***partial differentiation*** with respect to the ***time derivatives*** of the ***orientation angles*** of body B.

Using the same process, the ***body frame*** components of the ***partial angular velocity vectors*** can be written as a single matrix as follows.

$$\left[{}^{R}\omega'_{B,\dot{\theta}_B}\right]_{3\times 3} = \begin{bmatrix} C_{B2}C_{B3} & S_{B3} & 0 \\ -C_{B2}S_{B3} & C_{B3} & 0 \\ S_{B2} & 0 & 1 \end{bmatrix} \quad \text{(body frame components)} \qquad (5)$$

Here, $\left[{}^{R}\omega'_{B,\dot{\theta}_B}\right]$ is the ***partial angular velocity matrix*** of body B with respect to the ***angle derivatives*** expressed using ***body frame*** components. This matrix is the same as the coefficient matrix in Equation (3). A prime (i.e., "ω'") has been used to indicate ***body frame*** components.

Finally, using Equations (2) and (4), the ***fixed frame angular velocity*** components can be written in terms of the ***partial angular velocity matrix*** as follows.

$$\left\{\omega_B\right\} \triangleq \begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & S_{B2} \\ 0 & C_{B1} & -S_{B1}C_{B2} \\ 0 & S_{B1} & C_{B1}C_{B2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{B1} \\ \dot{\theta}_{B2} \\ \dot{\theta}_{B3} \end{Bmatrix} \triangleq \left[{}^{R}\omega_{B,\dot{\theta}_B}\right]\left\{\dot{\theta}_B\right\} \quad \text{(fixed frame components)} \qquad (6)$$

Similarly, using Equations (3) and (5), the ***body frame angular velocity*** components can be written in terms of the ***partial angular velocity matrix*** as follows.

$$\left\{\omega'_B\right\} \triangleq \begin{Bmatrix} \omega'_{B1} \\ \omega'_{B2} \\ \omega'_{B3} \end{Bmatrix} = \begin{bmatrix} C_{B2}C_{B3} & S_{B3} & 0 \\ -C_{B2}S_{B3} & C_{B3} & 0 \\ S_{B2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{B1} \\ \dot{\theta}_{B2} \\ \dot{\theta}_{B3} \end{Bmatrix} \triangleq \left[{}^{R}\omega'_{B,\dot{\theta}_B}\right]\left\{\dot{\theta}_B\right\} \quad \text{(body frame components)} \qquad (7)$$

Notes:

1. Because each ***column*** of the partial angular velocity matrices $\left[{}^{R}\omega_{B,\dot{\theta}_B}\right]$ and $\left[{}^{R}\omega'_{B,\dot{\theta}_B}\right]$ represent the ***components*** of ***partial angular velocity vectors***, the entries of the matrices depend on the ***choice*** of ***reference axes***. ***Fixed frame*** and ***body frame components*** are presented here.
2. The ***entries*** of the partial angular velocity matrices $\left[{}^{R}\omega_{B,\dot{\theta}_B}\right]$ and $\left[{}^{R}\omega'_{B,\dot{\theta}_B}\right]$ also depend on the ***orientation angle sequence***. Results for a 1-2-3 body fixed orientation angle sequence are presented here. Results for other body fixed orientation-angle sequences can be derived using the same process.

Partial Angular Velocities Using Angular Velocity Components as Generalized Speeds

Consider now using ***angular velocity*** components as the ***generalized speeds*** for a body B. Using ***fixed frame*** components of ${}^R\underset{\sim}{\omega}_B$ as the generalized speeds, the ***partial angular velocity vectors*** are as follows.

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_{B1}} = \underset{\sim}{N}_1} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_{B2}} = \underset{\sim}{N}_2} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_{B3}} = \underset{\sim}{N}_3}$$

and the partial angular velocity matrix is the 3×3 ***identity matrix***.

$$\boxed{\left[{}^R\omega_{B,\omega_B} \right]_{3\times 3} = \left[I \right]_{3\times 3} \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \text{(fixed frame components)} \qquad (8)$$

Using ***body frame*** components of ${}^R\underset{\sim}{\omega}_B$ as the generalized speeds, the ***partial angular velocity vectors*** are

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega'_{B1}} = \underset{\sim}{e}_1} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega'_{B2}} = \underset{\sim}{e}_2} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega'_{B3}} = \underset{\sim}{e}_3}$$

and the partial angular velocity matrix is again the 3×3 ***identity matrix***.

$$\boxed{\left[{}^R\omega'_{B,\omega'_B} \right]_{3\times 3} = \left[I \right]_{3\times 3} \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \text{(body frame components)} \qquad (9)$$

As above, the fixed frame and body frame components can be written in terms of these partial angular velocity matrices as follows.

$$\boxed{\left\{ \omega_B \right\} \triangleq \begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega_{B1} \\ \omega_{B2} \\ \omega_{B3} \end{Bmatrix} \triangleq \left[{}^R\omega_{B,\omega_B} \right] \left\{ \omega_B \right\}} \qquad \text{(fixed frame components)} \qquad (10)$$

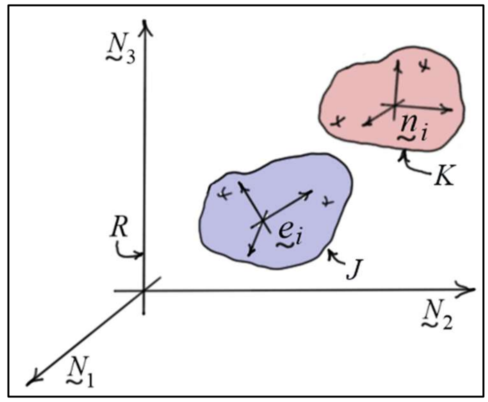
$$\boxed{\left\{ \omega'_B \right\} \triangleq \begin{Bmatrix} \omega'_{B1} \\ \omega'_{B2} \\ \omega'_{B3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega'_{B1} \\ \omega'_{B2} \\ \omega'_{B3} \end{Bmatrix} \triangleq \left[{}^R\omega'_{B,\omega'_B} \right] \left\{ \omega'_B \right\}} \qquad \text{(body frame components)} \qquad (11)$$

Notes:

1. Comparing Equations (6) and (7) with Equations (10) and (11), it is obvious that using ***angular velocity components*** as generalized speeds ***simplifies*** the partial angular velocity matrices.
2. The partial angular velocity matrices of Equations (10) and (11) are ***not dependent*** on which method is used to describe the orientation of the body. Any set of ***orientation angles*** or ***Euler parameters*** can be used.

Angular Velocity and Partial Angular Velocity Using Relative Coordinates

Angular Velocity Using a 1-2-3 Body fixed Rotation Sequence



Consider now the ***two-body system*** shown in the diagram. It may be convenient at times to express the ***angular motion*** of body K ***relative*** to another body in the system such as body J. To this end, let the angles $\theta_{Ji}\,(i=1,2,3)$ be the ***orientation angles*** of ***body*** J measured ***relative*** to the ***fixed frame*** R, and let the angles $\hat{\theta}_{Ki}\,(i=1,2,3)$ be the ***orientation angles*** of ***body*** K measured ***relative*** to ***body*** J. Here, the "hat" on the angle θ indicates the angles are measured relative to another body.

The ***reference frame*** in which the motion of a body is ***measured*** is referred to herein as the ***base frame*** of that body. So, the ***fixed frame*** is the ***base frame*** for ***body*** J, and the ***body*** J ***frame*** is the ***base frame*** for ***body*** K. The terms ***fixed frame***, ***base frame***, and ***body frame*** are used in the sequel.

Given that the ***body*** J ***frame*** is the ***base frame*** for ***body*** K, it is convenient to use the ***summation rule*** for ***angular velocities*** to find the angular velocity of body K relative to the fixed frame.

$$\boxed{{}^R\underset{\sim}{\omega}_K = {}^R\underset{\sim}{\omega}_J + {}^J\underset{\sim}{\omega}_K} \tag{12}$$

If the vectors ${}^R\underset{\sim}{\omega}_K$ and ${}^R\underset{\sim}{\omega}_J$ are written using ***fixed frame*** components and the vector ${}^J\underset{\sim}{\omega}_K$ is written using ***body*** J ***frame*** (or ***base frame***) components, then Equation (12) can be written in the following matrix form for the components.

$$\boxed{\left\{\omega_K\right\} = \left\{\omega_J\right\} + \left[R_J\right]^T \left\{{}^J\omega_K\right\} \triangleq \left\{\omega_J\right\} + \left[R_J\right]^T \left\{\hat{\omega}_K\right\}} \tag{13}$$

Here, $\left\{\omega_J\right\}$ and $\left\{\omega_K\right\}$ represent the ***fixed frame*** components of the ***angular velocities*** of bodies J and K relative to the ***fixed frame*** R, and $\left\{\hat{\omega}_K\right\}$ represents the ***body*** J components of the ***angular velocity*** of ***body*** K ***relative*** to ***body*** J. The transformation matrix $\left[R_J\right]^T$ ***converts body*** J components into ***fixed frame*** components.

As noted in Equation (2), when using a ***1-2-3 body fixed, orientation-angle sequence***, the base frame (fixed frame) components of the angular velocity of body J relative to the fixed frame can be written as follows.

$$\boxed{\left\{\omega_J\right\} = \begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} \text{(base frame (fixed frame) components)}} \tag{14}$$

Similarly, the base frame (body J frame) components of the ***angular velocity*** of body K ***relative*** to body J can be written as follows.

$$\left\{ {}^{J}\omega_{K} \right\} \triangleq \left\{ \hat{\omega}_{K} \right\} = \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad \text{(base frame (body } J \text{ frame) components)} \tag{15}$$

The ***transformation matrix*** $\left[R_J \right]$ that converts ***fixed frame*** components into ***body*** *J* frame components can be calculated as follows. Its transpose converts body *J* frame components into fixed frame components.

$$\left[R_J \right] = \left[{}^{R''}R_J \right] \left[{}^{R'}R_{R''} \right] \left[{}^{R}R_{R'} \right] = \begin{bmatrix} C_{J3} & S_{J3} & 0 \\ -S_{J3} & C_{J3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{J2} & 0 & -S_{J2} \\ 0 & 1 & 0 \\ S_{J2} & 0 & C_{J2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{J1} & S_{J1} \\ 0 & -S_{J1} & C_{J1} \end{bmatrix}$$

$$= \begin{bmatrix} C_{J3} & S_{J3} & 0 \\ -S_{J3} & C_{J3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{J2} & S_{J1}S_{J2} & -C_{J1}S_{J2} \\ 0 & C_{J1} & S_{J1} \\ S_{J2} & -S_{J1}C_{J2} & C_{J1}C_{J2} \end{bmatrix}$$

$$\Rightarrow \left[R_J \right] = \begin{bmatrix} C_{J2}C_{J3} & C_{J1}S_{J3} + S_{J1}S_{J2}C_{J3} & S_{J1}S_{J3} - C_{J1}S_{J2}C_{J3} \\ -C_{J2}S_{J3} & C_{J1}C_{J3} - S_{J1}S_{J2}S_{J3} & S_{J1}C_{J3} + C_{J1}S_{J2}S_{J3} \\ S_{J2} & -S_{J1}C_{J2} & C_{J1}C_{J2} \end{bmatrix} \tag{16}$$

The results in Equations (14) through (16) can now be substituted into the right side of Equation (13) to calculate the ***fixed frame components*** of ${}^{R}\underset{\sim}{\omega}_{K}$ the angular velocity of body *K* relative to the fixed frame.

Using this approach, the ***angular velocity*** components $\left\{ \omega_J \right\}$ of body *J* relative to the fixed frame *R* are expressed in the ***fixed frame***, and the ***angular velocity*** components $\left\{ \hat{\omega}_K \right\}$ of body *K* relative to body *J* are expressed in the ***body J frame***. In each case, the ***angular velocity*** components are expressed in the ***same frame*** in which the body ***orientation angles*** are ***measured***, that is, they are expressed in the ***base frames*** of the respective bodies.

Alternatively, the ***angular velocity*** components could be expressed in the ***same body frames***. For example, $\left\{ \omega'_K \right\}$ the ***body*** *K* components of ${}^{R}\underset{\sim}{\omega}_{K}$ can be written as follows.

$$\left\{ \omega'_K \right\} = \left[{}^{J}R_K \right] \left\{ \omega'_J \right\} + \left\{ \hat{\omega}'_K \right\} \tag{17}$$

Here, $\left\{ \omega'_J \right\}$ represents the ***body*** *J* components of the ***angular velocity*** of ***body*** *J* relative to the ***fixed frame*** *R*, and $\left\{ \hat{\omega}'_K \right\}$ represents the ***body*** *K* components of the ***angular velocity*** of ***body*** *K* ***relative*** to ***body*** *J*. The transformation matrix $\left[{}^{J}R_K \right]$ ***converts body*** *J* ***components*** into ***body*** *K* ***components***.

As noted in Equation (3), when using a ***1-2-3 body fixed, orientation-angle sequence***, the body J components of the angular velocity of body J relative to the fixed frame can be written as follows.

$$\left\{\omega'_J\right\} = \begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} \quad \text{(body } J \text{ frame components)} \qquad (18)$$

Similarly, the ***body*** K ***frame*** components of the ***angular velocity*** of body K ***relative*** to body J can be written as follows.

$$\left\{\hat{\omega}'_K\right\} = \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad \text{(body } K \text{ frame components)} \qquad (19)$$

The ***transformation matrix*** that converts ***body*** J ***frame*** components into ***body*** K frame components can be calculated as follows.

$$\left[{}^{J}R_K\right] = \left[{}^{R''}R_K\right]\left[{}^{R'}R_{R''}\right]\left[{}^{R}R_{R'}\right] = \begin{bmatrix} C_{K3} & S_{K3} & 0 \\ -S_{K3} & C_{K3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{K2} & 0 & -S_{K2} \\ 0 & 1 & 0 \\ S_{K2} & 0 & C_{K2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{K1} & S_{K1} \\ 0 & -S_{K1} & C_{K1} \end{bmatrix}$$

$$= \begin{bmatrix} C_{K3} & S_{K3} & 0 \\ -S_{K3} & C_{K3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{K2} & S_{K1}S_{K2} & -C_{K1}S_{K2} \\ 0 & C_{K1} & S_{K1} \\ S_{K2} & -S_{K1}C_{K2} & C_{K1}C_{K2} \end{bmatrix}$$

$$\Rightarrow \left[{}^{J}R_K\right] = \begin{bmatrix} C_{K2}C_{K3} & C_{K1}S_{K3} + S_{K1}S_{K2}C_{K3} & S_{K1}S_{K3} - C_{K1}S_{K2}C_{K3} \\ -C_{K2}S_{K3} & C_{K1}C_{K3} - S_{K1}S_{K2}S_{K3} & S_{K1}C_{K3} + C_{K1}S_{K2}S_{K3} \\ S_{K2} & -S_{K1}C_{K2} & C_{K1}C_{K2} \end{bmatrix} \qquad (20)$$

Using this approach, the components of ${}^{R}\underset{\sim}{\omega}_J$ the ***angular velocity*** of ***body*** J relative to the fixed frame R are resolved in the ***body*** J ***frame***, and the components of ${}^{R}\underset{\sim}{\omega}_K$ the angular velocity of body K relative to the fixed frame are resolved in the ***body*** K ***frame***. The components of ${}^{J}\underset{\sim}{\omega}_K$ the ***angular velocity*** of ***body*** K ***relative*** to J are also resolved in the ***body*** K ***frame***. In each case, the angular velocity components of body J and body K are resolved in their respective ***body frames***.

Notes:

1. ***Relative coordinates*** are often used because the ***motions*** between ***adjoining bodies*** of a system are more ***naturally*** described in terms of ***relative coordinates***.
2. Unfortunately, the ***equations*** associated with the ***kinematics*** of the system are usually ***more complex*** when written in terms of ***relative coordinates***.

Partial Angular Velocities Using Orientation Angle Derivatives as Generalized Speeds

Using Equations (13), (14), and (15), the ***fixed frame*** components of the ***partial angular velocity matrices*** for each of the two bodies can be written as

$$\left[{}^R\omega_{J,\dot{\theta}_J}\right] = \begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix} \qquad \left[{}^R\omega_{J,\dot{\hat{\theta}}_K}\right] = \left[\,0\,\right]_{3\times 3} \qquad \text{(fixed frame components)} \qquad (21)$$

$$\left[{}^R\omega_{K,\dot{\theta}_J}\right] = \left[{}^R\omega_{J,\dot{\theta}_J}\right] \qquad \left[{}^R\omega_{K,\dot{\hat{\theta}}_K}\right] = \left[R_J\right]^T \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \qquad \text{(fixed frame components)} \qquad (22)$$

Using Equations (21) and (22), the ***fixed frame*** components of the ***angular velocities*** of the two bodies can be written as follows.

$$\left\{\omega_J\right\} \triangleq \begin{Bmatrix} \omega_{J1} \\ \omega_{J2} \\ \omega_{J3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \qquad \text{(fixed frame components)} \qquad (23)$$

$$\triangleq \left[{}^R\omega_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^R\omega_{J,\dot{\hat{\theta}}_K}\right]\left\{\dot{\hat{\theta}}_K\right\}$$

$$\left\{\omega_K\right\} \triangleq \begin{Bmatrix} \omega_{K1} \\ \omega_{K2} \\ \omega_{K3} \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} + \left[R_J\right]^T \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \qquad \text{(fixed frame components)} \qquad (24)$$

$$\triangleq \left[{}^R\omega_{K,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^R\omega_{K,\dot{\hat{\theta}}_K}\right]\left\{\dot{\hat{\theta}}_K\right\}$$

$$= \left[{}^R\omega_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^R\omega_{K,\dot{\hat{\theta}}_K}\right]\left\{\dot{\hat{\theta}}_K\right\}$$

Using Equations (17), (18), and (19), the ***body frame*** components of the ***partial angular velocity matrices*** for each of the two bodies can be written as follows.

$$\left[{}^R\omega'_{J,\dot{\theta}_J}\right] = \begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix} \qquad \left[{}^R\omega'_{J,\dot{\hat{\theta}}_K}\right] = \left[\,0\,\right]_{3\times 3} \qquad \text{(body } J \text{ components)} \qquad (25)$$

$$\left[{}^R\omega'_{K,\dot{\theta}_J}\right]=\left[{}^J R_K\right]\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix} \quad \left[{}^R\omega'_{K,\dot{\hat{\theta}}_K}\right]=\begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix} \quad \text{(body } K \text{ components)} \quad (26)$$

Using Equations (25) and (26), the ***same-body*** components of the ***angular velocities*** of the two bodies can be written as follows.

$$\left\{\omega'_J\right\} \triangleq \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} = \begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad \text{(body } J \text{ components)} \quad (27)$$

$$\triangleq \left[{}^R\omega'_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^R\omega'_{J,\dot{\hat{\theta}}_K}\right]\left\{\dot{\hat{\theta}}_K\right\}$$

$$\left\{\omega'_K\right\} \triangleq \begin{Bmatrix} \omega'_{K1} \\ \omega'_{K2} \\ \omega'_{K3} \end{Bmatrix} = \left[{}^J R_K\right]\left[\omega'_{J,\dot{\theta}_J}\right]\begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} + \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad \text{(body } K \text{ components)} \quad (28)$$

$$\triangleq \left[{}^R\omega'_{K,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^R\omega'_{K,\dot{\hat{\theta}}_K}\right]\left\{\dot{\hat{\theta}}_K\right\}$$

Extension to Multiple Body Systems

The process described above can be ***extended*** to systems with many bodies. To do this, consider bodies J and K to be two bodies ***within*** a ***larger system*** with $J = \mathfrak{L}(K)$, that is, with body J as the ***lower-numbered body*** of body K. Next, for a system of N bodies, define the ***system column vector*** of ***relative angles*** as follows.

$$\{\theta\}_{3N\times1} = \begin{bmatrix} \hat{\theta}_{11} & \hat{\theta}_{12} & \hat{\theta}_{13} & \cdots & \hat{\theta}_{J1} & \hat{\theta}_{J2} & \hat{\theta}_{J3} & \cdots & \hat{\theta}_{K1} & \hat{\theta}_{K2} & \hat{\theta}_{K3} & \cdots & \hat{\theta}_{N1} & \hat{\theta}_{N2} & \hat{\theta}_{N3} \end{bmatrix}^T \quad (29)$$

Each set of ***three angles*** describes the ***orientation*** of a body ***relative*** to its ***lower numbered body***. The first set of angles describes the orientation of body *1* (system reference body) ***relative*** to the ***fixed frame***.

Then, using Equation (13) with ***base frame components*** of the ***relative angular velocity vectors***, write the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body K as follows.

$$\left\{\omega_K\right\} = \left[{}^R\omega_{K,\dot{\theta}}\right]\left\{\dot{\theta}\right\} = \left\{\omega_J\right\} + \left[R_J\right]^T\left\{\hat{\omega}_K\right\} = \left[{}^R\omega_{J,\dot{\theta}}\right]\left\{\dot{\theta}\right\} + \left[R_J\right]^T\begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix}\begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad (30)$$

Note that ${}^R\underset{\sim}{\omega}_J$ the angular velocity of body J ***does not depend*** on $\dot{\hat{\theta}}_{Ki}$ $(i=1,2,3)$, because body J is the lower-numbered body of body K. So, $\left[{}^R\omega_{K,\dot{\theta}}\right]_{3\times3N}$ the partial angular velocity matrix of body K can be ***built*** as follows.

1. First, set

$$\left[{}^{R}\omega_{K,\dot{\theta}}\right]_{3\times 3N}=\left[{}^{R}\omega_{J,\dot{\theta}}\right]_{3\times 3N} \tag{31}$$

2. Then, set the ***three columns*** associated with $\dot{\hat{\theta}}_{Ki}\ (i=1,2,3)$ as follows.

$$\left[{}^{R}\omega_{K,\dot{\theta}}\right]_{ik}=\left[\left[R_J\right]^T\begin{bmatrix}1 & 0 & S_{K2}\\ 0 & C_{K1} & -S_{K1}C_{K2}\\ 0 & S_{K1} & C_{K1}C_{K2}\end{bmatrix}\right]_{ij}\quad (i=1,2,3;\ j=1,2,3;\ k=3K-3+j) \tag{32}$$

For body *1*, only Equation (32) applies giving the following result.

$$\left[{}^{R}\omega_{K,\dot{\theta}}\right]_{ij}=\begin{bmatrix}1 & 0 & S_{K2}\\ 0 & C_{K1} & -S_{K1}C_{K2}\\ 0 & S_{K1} & C_{K1}C_{K2}\end{bmatrix}_{ij}\quad (i=1,2,3;\ j=1,2,3;\ K=1)$$

All other entries are zero.

Using Equation (17) with ***body frame components*** of the ***relative angular velocity vectors***, write the ***body frame components*** of ${}^{R}\underset{\sim}{\omega}_K$ the angular velocity of body K as follows.

$$\left\{\omega'_K\right\}=\left[{}^{R}\omega'_{K,\dot{\theta}}\right]\left\{\dot{\theta}\right\}=\left[{}^{J}R_K\right]\left\{\omega'_J\right\}+\left\{\hat{\omega}'_K\right\}=\left[{}^{J}R_K\right]\left[{}^{R}\omega'_{J,\dot{\theta}}\right]\left\{\dot{\theta}\right\}+\begin{bmatrix}C_{K2}C_{K3} & S_{K3} & 0\\ -C_{K2}S_{K3} & C_{K3} & 0\\ S_{K2} & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\hat{\theta}}_{K1}\\ \dot{\hat{\theta}}_{K2}\\ \dot{\hat{\theta}}_{K3}\end{Bmatrix} \tag{33}$$

Noting again that ${}^{R}\underset{\sim}{\omega}_J$ the ***angular velocity*** of body J ***does not depend*** on $\dot{\hat{\theta}}_{Ki}\ (i=1,2,3)$, $\left[{}^{R}\omega'_{K,\dot{\theta}}\right]_{3\times 3N}$ the partial angular velocity matrix of body K can be ***built*** as follows.

1. First, set

$$\left[{}^{R}\omega'_{K,\dot{\theta}}\right]_{3\times 3N}=\left[\left[{}^{J}R_K\right]\left[{}^{R}\omega'_{J,\dot{\theta}}\right]\right]_{3\times 3N} \tag{34}$$

2. Then, set the ***three columns*** associated with $\dot{\hat{\theta}}_{Ki}\ (i=1,2,3)$ as follows.

$$\left[{}^{R}\omega'_{K,\dot{\theta}}\right]_{ik}=\begin{bmatrix}C_{K2}C_{K3} & S_{K3} & 0\\ -C_{K2}S_{K3} & C_{K3} & 0\\ S_{K2} & 0 & 1\end{bmatrix}_{ij}\quad (i=1,2,3;\ j=1,2,3;\ k=3K-3+j) \tag{35}$$

Again, for body *1*, only Equation (35) applies giving the following result.

$$\left[{}^{R}\omega'_{K,\dot{\theta}}\right]_{ij}=\begin{bmatrix}C_{K2}C_{K3} & S_{K3} & 0\\ -C_{K2}S_{K3} & C_{K3} & 0\\ S_{K2} & 0 & 1\end{bmatrix}_{ij}\quad (i=1,2,3;\ j=1,2,3;\ K=1)$$

All other entries are zero.

Partial Angular Velocities Using Angular Velocity Components as Generalized Speeds

Using Equation (13), the ***fixed frame*** components of the ***partial angular velocity matrices*** for each of the two bodies can be written as

$$\left[{}^{R}\omega_{J,\omega_J}\right]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad\left[{}^{R}\omega_{J,\hat{\omega}_K}\right]=\left[\;0\;\right]_{3\times3}\qquad\text{(fixed frame components)}\qquad(36)$$

$$\left[{}^{R}\omega_{K,\omega_J}\right]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad\left[{}^{R}\omega_{K,\hat{\omega}_K}\right]=\left[R_J\right]^T\qquad\text{(fixed frame components)}\qquad(37)$$

Using Equations (36) and (37), the ***fixed frame angular velocity*** components of the two bodies can be written as follows.

$$\begin{aligned}\left\{\omega_J\right\}&\triangleq\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}\\&\triangleq\left[{}^{R}\omega_{J,\omega_J}\right]\left\{\omega_J\right\}+\left[{}^{R}\omega_{J,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\}\end{aligned}\qquad\text{(fixed frame components)}\qquad(38)$$

$$\begin{aligned}\left\{\omega_K\right\}&\triangleq\begin{Bmatrix}\omega_{K1}\\\omega_{K2}\\\omega_{K3}\end{Bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}+\left[R_J\right]^T\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}\\&\triangleq\left[{}^{R}\omega_{K,\omega_J}\right]\left\{\omega_J\right\}+\left[{}^{R}\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\}\\&=\left[{}^{R}\omega_{J,\omega_J}\right]\left\{\omega_J\right\}+\left[{}^{R}\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\}\end{aligned}\qquad\text{(fixed frame components)}\qquad(39)$$

Using Equation (17), the ***same-body*** components of the ***partial angular velocity matrices*** for each of the bodies can be written as

$$\left[{}^{R}\omega'_{J,\omega'_J}\right]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad\left[{}^{R}\omega'_{J,\hat{\omega}'_K}\right]=\left[\;0\;\right]_{3\times3}\qquad\text{(body } J \text{ components)}\qquad(40)$$

$$\left[{}^{R}\omega'_{K,\omega'_J}\right]=\left[{}^{J}R_K\right]\qquad\left[{}^{R}\omega'_{K,\hat{\omega}'_K}\right]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad\text{(body } K \text{ components)}\qquad(41)$$

Using Equations (40) and (41), the ***same body*** components of the ***angular velocities*** of the two bodies can be written as follows.

$$\left\{\omega'_J\right\} \triangleq \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} \triangleq \left[{}^R\omega_{J,\omega'_J} \right] \left\{\omega'_J\right\} + \left[{}^R\omega_{J,\hat{\omega}'_K} \right] \left\{\hat{\omega}'_K\right\} \quad \text{(body } J \text{ components)} \qquad (42)$$

$$\left\{\omega'_K\right\} \triangleq \begin{Bmatrix} \omega'_{K1} \\ \omega'_{K2} \\ \omega'_{K3} \end{Bmatrix} = \left[{}^J R_K \right] \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} \triangleq \left[{}^R\omega'_{K,\omega'_J} \right] \left\{\omega'_J\right\} + \left[{}^R\omega_{K,\hat{\omega}'_K} \right] \left\{\hat{\omega}'_K\right\} \quad \text{(body } K \text{ components)} \qquad (43)$$

Extension to Multiple Body Systems

The process described above can be ***extended*** to systems with many bodies. To do this, consider bodies *J* and *K* to be two bodies ***within*** a ***larger system*** with $J = \mathcal{L}(K)$, that is, with body *J* as the ***lower numbered body*** of body *K*. When using ***base frame relative angular velocity components*** for a system of *N* bodies, define the ***system column vector*** of ***relative angular velocity components*** as follows.

$$\{\omega\}_{3N\times 1} = \begin{bmatrix} \hat{\omega}_{11} & \hat{\omega}_{12} & \hat{\omega}_{13} & \cdots & \hat{\omega}_{J1} & \hat{\omega}_{J2} & \hat{\omega}_{J3} & \cdots & \hat{\omega}_{K1} & \hat{\omega}_{K2} & \hat{\omega}_{K3} & \cdots & \hat{\omega}_{N1} & \hat{\omega}_{N2} & \hat{\omega}_{N3} \end{bmatrix}^T \qquad (44)$$

When using ***body frame relative angular velocity components*** for a system of *N* bodies, define the ***system column vector*** of ***relative angular velocity components*** as follows.

$$\{\omega\}_{3N\times 1} = \begin{bmatrix} \hat{\omega}'_{11} & \hat{\omega}'_{12} & \hat{\omega}'_{13} & \cdots & \hat{\omega}'_{J1} & \hat{\omega}'_{J2} & \hat{\omega}'_{J3} & \cdots & \hat{\omega}'_{K1} & \hat{\omega}'_{K2} & \hat{\omega}'_{K3} & \cdots & \hat{\omega}'_{N1} & \hat{\omega}'_{N2} & \hat{\omega}'_{N3} \end{bmatrix}^T \qquad (45)$$

Each set of ***three components*** in the two column vectors describes the ***angular velocity*** of a body ***relative*** to its ***lower numbered body***. The first set of components describes the angular velocity of body *1* (system reference body) ***relative*** to the ***fixed frame***.

Using Equation (13) with ***base frame components*** of the ***relative angular velocity vectors***, write the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body *K* as follows.

$$\left\{\omega_K\right\} = \left[{}^R\omega_{K,\omega} \right] \{\omega\} = \left\{\omega_J\right\} + \left[R_J\right]^T \left\{\hat{\omega}_K\right\} = \left[{}^R\omega_{J,\omega} \right] \{\omega\} + \left[R_J\right]^T \left\{\hat{\omega}_K\right\} \qquad (46)$$

Note that ${}^R\underset{\sim}{\omega}_J$ the angular velocity of body *J* ***does not depend*** on $\hat{\omega}_{Ki}\ (i = 1,2,3)$, because body *J* is the ***lower numbered*** body of body *K*. So, $\left[{}^R\omega_{K,\omega} \right]_{3\times 3N}$ the partial angular velocity matrix of body *K* can be ***built*** as follows.

1. First, set

$$\left[{}^R\omega_{K,\omega} \right]_{3\times 3N} = \left[{}^R\omega_{J,\omega} \right]_{3\times 3N} \qquad (47)$$

2. Then, set the ***three columns*** associated with $\hat{\omega}_{Ki}\ (i = 1,2,3)$ as follows.

$$\left[{}^{R}\omega_{K,\omega}\right]_{ik} = \left[R_J\right]^{T}_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3K-3+j\right) \tag{48}$$

For body *1*, only Equation (48) applies giving the following result.

$$\left[{}^{R}\omega_{K,\omega}\right]_{ij} = \left[I\right]_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ K=1\right)$$

All other entries are zero.

Using Equation (17) with ***body frame components*** of the ***relative angular velocity vectors***, write the ***body frame components*** of ${}^{R}\underset{\sim}{\omega}_K$ the angular velocity of body K as follows.

$$\left\{\omega'_K\right\} = \left[{}^{R}\omega'_{K,\omega}\right]\left\{\omega\right\} = \left[{}^{J}R_K\right]\left\{\omega'_J\right\} + \left\{\hat{\omega}'_K\right\} = \left[{}^{J}R_K\right]\left[{}^{R}\omega'_{J,\omega}\right]\left\{\omega\right\} + \left\{\hat{\omega}'_K\right\} \tag{49}$$

Noting again that ${}^{R}\underset{\sim}{\omega}_J$ the angular velocity of body J ***does not depend*** on $\hat{\omega}'_{Ki}$ $\left(i=1,2,3\right)$, $\left[{}^{R}\omega'_{K,\omega}\right]_{3\times 3N}$ the partial angular velocity matrix of body K can be ***built*** as follows.

1. First, set

$$\left[{}^{R}\omega'_{K,\omega}\right]_{3\times 3N} = \left[\left[{}^{J}R_K\right]\left[{}^{R}\omega'_{J,\omega}\right]\right]_{3\times 3N} \tag{50}$$

2. Then, set the ***three columns*** associated with $\hat{\omega}'_{Ki}$ $\left(i=1,2,3\right)$ as follows.

$$\left[{}^{R}\omega'_{K,\omega}\right]_{ik} = \left[I\right]_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3K-3+j\right) \tag{51}$$

   Here, $\left[I\right]$ is the 3×3 identity matrix.

Again, for body *1*, only Equation (51) applies giving the following result.

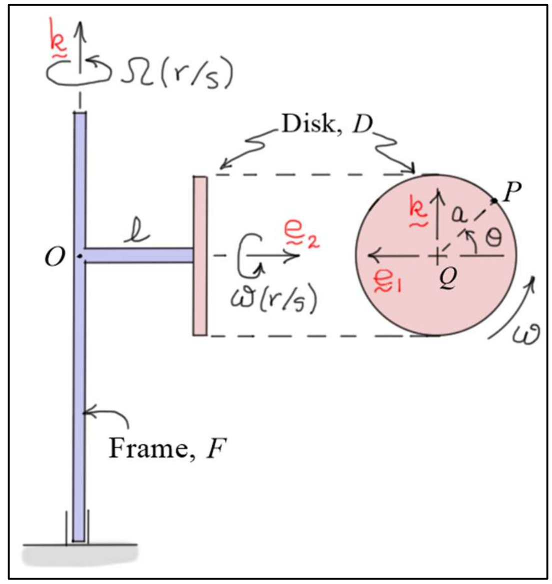
$$\left[{}^{R}\omega'_{K,\omega}\right]_{ij} = \left[I\right]_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ K=1\right)$$

All other entries are zero.

Examples

Example 1

The system shown consists of two connected bodies – the vertical frame F and the disk D. Frame F rotates at a rate of $\dot{\phi} = \Omega$ (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed to and rotates relative to F at a rate of $\dot{\theta} = \omega$ (rad/s) about the horizontal arm of F (direction annotated by the rotating unit vector $\underset{\sim}{e}_2$).



Reference frames: (all frames align when $\phi = \theta = 0$)

$R: (\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k})$	(fixed frame)
$F: (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$	(rotating with frame F)
$D: (\underset{\sim}{n}_1, \underset{\sim}{e}_2, \underset{\sim}{n}_3)$	(rotating with disk D)

Complete the following. Use $\{\beta\}$ as the column matrix of angles ϕ and θ. Expressing all results in matrix form.

a) Find $\{\omega_D\}$ the ***fixed frame components*** and of the ***angular velocity*** of disk D in R and $\left[{}^R\omega_{D,\dot{\beta}}\right]$ the matrix of ***fixed frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Express the results in terms of the angles ϕ, θ, and their time derivatives.

b) Find $\{\omega'_D\}$ the ***disk frame components*** of the ***angular velocity*** of disk D in R and $\left[{}^R\omega'_{D,\dot{\beta}}\right]$ the matrix of ***disk frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Express the results in terms of the angles ϕ, θ, and their time derivatives.

Solution:

a) The ***fixed frame components*** of the ***angular velocity*** of D in the fixed frame R can be written as follows.

$$\{\omega_D\} = \{\dot{\phi}'\} + \left[R_F\right]^T \{\dot{\theta}'\} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -S_\phi \dot{\theta} \\ C_\phi \dot{\theta} \\ \dot{\phi} \end{Bmatrix} = \begin{bmatrix} 0 & -S_\phi \\ 0 & C_\phi \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \end{Bmatrix}$$

$$\Rightarrow \{\omega_D\} = \begin{bmatrix} 0 & -S_\phi \\ 0 & C_\phi \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \end{Bmatrix} \triangleq \left[{}^R\omega_{D,\dot{\beta}}\right]\{\dot{\beta}\} \tag{52}$$

b) The ***body frame components*** of the ***angular velocity*** of D in the fixed frame R can be written as follows.

$$\{\omega'_D\} = \left[R_D\right]\{\dot{\phi}'\} + \{\dot{\theta}'\} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix}$$

Or,

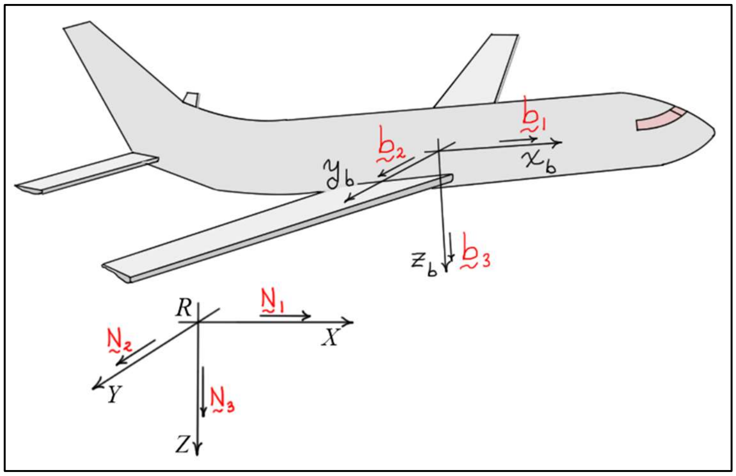
$$\left\{\omega'_D\right\}=\left[{}^F R_D\right]\left\{\dot{\phi}'\right\}+\left\{\dot{\theta}'\right\}=\begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix}+\begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{\left\{\omega'_D\right\}=\begin{Bmatrix} -S_\theta\dot{\phi} \\ \dot{\theta} \\ C_\theta\dot{\phi} \end{Bmatrix}=\begin{bmatrix} -S_\theta & 0 \\ 0 & 1 \\ C_\theta & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \end{Bmatrix}\triangleq\left[{}^R\omega'_{D,\dot{\beta}}\right]\left\{\dot{\beta}\right\}} \qquad (53)$$

Note that the rotation of *F* in the fixed frame *R* ***does not alter*** the results of Equation (53), because the unit vector $\underset{\sim}{k}$ is fixed in ***both*** the rotating frame *F* and the fixed frame *R*.

Example 2

The orientation of an aircraft *A* can be defined using a 3-2-1 body fixed rotation sequence. As before, the body axes $A:\left(\underset{\sim}{b}_1,\underset{\sim}{b}_2,\underset{\sim}{b}_3\right)$ are initially aligned with the fixed frame axes $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$. It is common to refer to these angles as ψ, θ, and ϕ. For small angles they are equivalent to the "yaw", "pitch", and "roll" angles of the aircraft. Complete the following expressing all results in matrix form. Use $\{\beta\}$ as the column matrix of angles ψ, θ, and ϕ.



a) Find $\left\{\omega_A\right\}$ the ***fixed frame components*** of the ***angular velocity*** of *A* relative to the fixed frame *R* and $\left[{}^R\omega_{A,\dot{\beta}}\right]$ the matrix of ***fixed frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Express the results in terms of the angles ψ, θ, ϕ, and their time derivatives.

b) Find $\left\{\omega'_A\right\}$ the ***body frame components*** of the ***angular velocity*** of *A* relative to the fixed frame *R* and $\left[{}^R\omega'_{A,\dot{\beta}}\right]$ the matrix of ***body frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Express the results in terms of the angles ψ, θ, ϕ, and their time derivatives.

Solution:

a) Given a 3-2-1 body fixed rotation sequence, the ***angular velocity*** of the aircraft can be written as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_A=\dot{\psi}\,\underset{\sim}{N}_3+\dot{\theta}\,\underset{\sim}{N}'_2+\dot{\phi}\,\underset{\sim}{N}''_1}$$

In matrix notation, the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_A$ can be calculated as follows.

$$\left\{\omega_A\right\}=\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix}+\left[{}^R R_{R'}\right]^T\begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}+\left(\left[{}^{R'}R_{R''}\right]\left[{}^R R_{R'}\right]\right)^T\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix}+\left[{}^R R_{R'}\right]^T\begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}+\left[{}^R R_{R'}\right]^T\left[{}^{R'}R_{R''}\right]^T\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}$$

$$=\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix}+\begin{bmatrix}C_\psi & -S_\psi & 0\\S_\psi & C_\psi & 0\\0 & 0 & 1\end{bmatrix}\begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}+\begin{bmatrix}C_\psi & -S_\psi & 0\\S_\psi & C_\psi & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}C_\theta & 0 & S_\theta\\0 & 1 & 0\\-S_\theta & 0 & C_\theta\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}$$

$$=\begin{Bmatrix}-S_\psi\dot{\theta}\\C_\psi\dot{\theta}\\\dot{\psi}\end{Bmatrix}+\begin{bmatrix}C_\psi & -S_\psi & 0\\S_\psi & C_\psi & 0\\0 & 0 & 1\end{bmatrix}\begin{Bmatrix}C_\theta\dot{\phi}\\0\\-S_\theta\dot{\phi}\end{Bmatrix}$$

$$\Rightarrow\boxed{\left\{\omega_A\right\}=\begin{Bmatrix}-S_\psi\dot{\theta}+C_\psi C_\theta\dot{\phi}\\C_\psi\dot{\theta}+S_\psi C_\theta\dot{\phi}\\\dot{\psi}-S_\theta\dot{\phi}\end{Bmatrix}}\quad\text{(fixed frame components)}$$

Here, the reference frames R' and R'' represent the ***intermediate reference frames*** defined as part of the 3-2-1 rotation sequence. Hence, matrix $\left[{}^R R_{R'}\right]$ represents a transformation matrix associated with a "3" rotation, and matrix $\left[{}^{R'}R_{R''}\right]$ represents a transformation matrix associated with a "2" rotation. See the development of Equations (2) and (3) above. Using this result, the ***fixed frame components*** of the partial angular velocity matrix can then be identified as follows.

$$\boxed{\left\{\omega_A\right\}=\begin{Bmatrix}-S_\psi\dot{\theta}+C_\psi C_\theta\dot{\phi}\\C_\psi\dot{\theta}+S_\psi C_\theta\dot{\phi}\\\dot{\psi}-S_\theta\dot{\phi}\end{Bmatrix}=\begin{bmatrix}0 & -S_\psi & C_\psi C_\theta\\0 & C_\psi & S_\psi C_\theta\\1 & 0 & -S_\theta\end{bmatrix}\begin{Bmatrix}\dot{\psi}\\\dot{\theta}\\\dot{\phi}\end{Bmatrix}\triangleq\left[{}^R\omega_{A,\dot{\beta}}\right]\left\{\dot{\beta}\right\}}\quad\text{(fixed frame components)}\quad(54)$$

b) Given a 3-2-1 body fixed rotation sequence, the ***angular velocity*** of the ***aircraft*** can also be written as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_A=\dot{\psi}\,\underset{\sim}{N}'_3+\dot{\theta}\,\underset{\sim}{N}''_2+\dot{\phi}\,\underset{\sim}{b}_1}$$

In matrix notation, the ***body frame components*** of ${}^R\underset{\sim}{\omega}_A$ can be calculated as follows.

$$\left\{\omega'_A\right\}=\left[{}^{R''}R_A\right]\left[{}^{R'}R_{R''}\right]\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix}+\left[{}^{R''}R_A\right]\begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}+\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\0 & C_\phi & S_\phi\\0 & -S_\phi & C_\phi\end{bmatrix}\begin{bmatrix}C_\theta & 0 & -S_\theta\\0 & 1 & 0\\S_\theta & 0 & C_\theta\end{bmatrix}\begin{Bmatrix}0\\0\\\dot{\psi}\end{Bmatrix}+\begin{bmatrix}1 & 0 & 0\\0 & C_\phi & S_\phi\\0 & -S_\phi & C_\phi\end{bmatrix}\begin{Bmatrix}0\\\dot{\theta}\\0\end{Bmatrix}+\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & C_\phi & S_\phi\\ 0 & -S_\phi & C_\phi\end{bmatrix}\begin{Bmatrix}-S_\theta\dot{\psi}\\ 0\\ C_\theta\dot{\psi}\end{Bmatrix}+\begin{Bmatrix}\dot{\phi}\\ C_\phi\dot{\theta}\\ -S_\phi\dot{\theta}\end{Bmatrix}$$

$$\Rightarrow \boxed{\left\{\omega'_A\right\}=\begin{Bmatrix}-S_\theta\dot{\psi}+\dot{\phi}\\ C_\theta S_\phi\dot{\psi}+C_\phi\dot{\theta}\\ C_\theta C_\phi\dot{\psi}-S_\phi\dot{\theta}\end{Bmatrix}} \quad \text{(body frame components)}$$

Here, matrix $\left[{}^{R'}R_{R''}\right]$ represents a transformation matrix associated with a "2" rotation, and matrix $\left[{}^{R''}R_A\right]$ represents a transformation matrix associated with a "1" rotation. Using this result, the ***body frame components*** of the ***partial angular velocity matrix*** can be identified as follows.

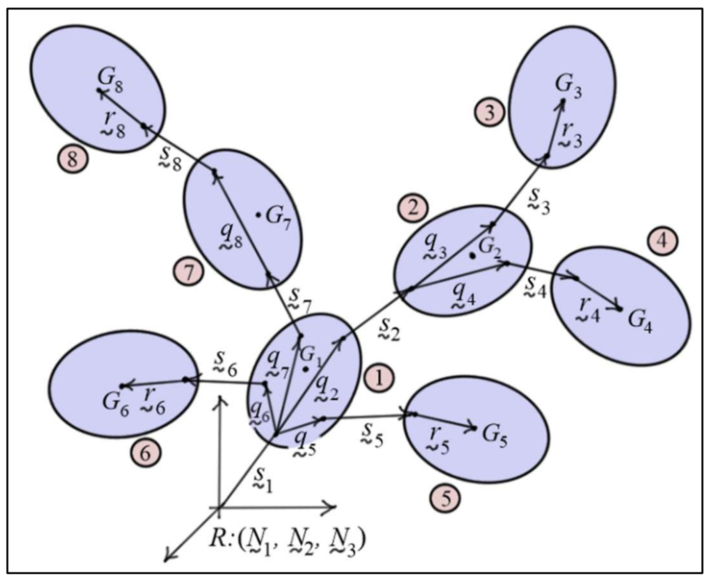
$$\boxed{\left\{\omega'_A\right\}=\begin{Bmatrix}-S_\theta\dot{\psi}+\dot{\phi}\\ C_\theta S_\phi\dot{\psi}+C_\phi\dot{\theta}\\ C_\theta C_\phi\dot{\psi}-S_\phi\dot{\theta}\end{Bmatrix}=\begin{bmatrix}-S_\theta & 0 & 1\\ C_\theta S_\phi & C_\phi & 0\\ C_\theta C_\phi & -S_\phi & 0\end{bmatrix}\begin{Bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{Bmatrix}\triangleq\left[{}^R\omega'_{A,\dot{\beta}}\right]\left\{\dot{\beta}\right\}} \quad \text{(body frame components)} \quad (55)$$

Example 3

The figure shows an eight-body system numbered using the numbering scheme presented in Unit 1. Body 1 is the system reference body, and the rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower-numbered body array for the system is as follows.

$$\mathcal{L}(1,\ldots,8)=(0,1,2,2,1,1,1,7)$$

The orientation of body 1 is defined relative to the fixed frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, and the orientations of all the other bodies are defined relative to their adjacent, lower-numbered bodies. Using ***base frame components*** of the ***relative angular velocities*** of the bodies as generalized speeds, complete the following.



a) Define the ***fixed frame components*** of the ***angular velocities*** for all bodies in the system.

b) Combine the ***relative angular velocity components*** into a single 24×1 ***system matrix*** $\{\omega\}_{24\times 1}$.

c) Define the ***fixed frame components*** of the ***partial angular velocities*** for all the bodies in the system.

d) Define a 3×24 ***partial angular velocity matrix*** for ***each body*** in the system.

e) Write the ***fixed frame components*** of the ***angular velocity*** of each body in terms of the ***system angular velocity matrix*** defined in part (b) and the ***partial angular velocity matrices*** defined in part (d).

Solution:

a) $\{\omega_K\}$ $(K=1,\ldots,8)$ are 3×1 vectors of the ***fixed frame components*** of the angular velocities of the bodies.

$\{\hat{\omega}_K\}$ $(K=1,\ldots,8)$ are 3×1 vectors of the ***base frame components*** of the angular velocities of the bodies ***relative*** to their ***base frames*** (lower-numbered bodies).

$${}^R\underset{\sim}{\omega}_1 = \hat{\underset{\sim}{\omega}}_1 \qquad \boxed{\{\omega_1\} = \{\hat{\omega}_1\} = \begin{bmatrix}\hat{\omega}_{11} & \hat{\omega}_{12} & \hat{\omega}_{13}\end{bmatrix}^T}$$

$${}^R\underset{\sim}{\omega}_2 = {}^R\underset{\sim}{\omega}_1 + \hat{\underset{\sim}{\omega}}_2 \qquad \boxed{\{\omega_2\} = \{\omega_1\} + [R_1]^T\{\hat{\omega}_2\}}$$

$${}^R\underset{\sim}{\omega}_3 = {}^R\underset{\sim}{\omega}_2 + \hat{\underset{\sim}{\omega}}_3 \qquad \boxed{\{\omega_3\} = \{\omega_2\} + [R_2]^T\{\hat{\omega}_3\}}$$

$${}^R\underset{\sim}{\omega}_4 = {}^R\underset{\sim}{\omega}_2 + \hat{\underset{\sim}{\omega}}_4 \qquad \boxed{\{\omega_4\} = \{\omega_2\} + [R_2]^T\{\hat{\omega}_4\}}$$

$${}^R\underset{\sim}{\omega}_5 = {}^R\underset{\sim}{\omega}_1 + \hat{\underset{\sim}{\omega}}_5 \qquad \boxed{\{\omega_5\} = \{\omega_1\} + [R_1]^T\{\hat{\omega}_5\}}$$

$${}^R\underset{\sim}{\omega}_6 = {}^R\underset{\sim}{\omega}_1 + \hat{\underset{\sim}{\omega}}_6 \qquad \boxed{\{\omega_6\} = \{\omega_1\} + [R_1]^T\{\hat{\omega}_6\}}$$

$${}^R\underset{\sim}{\omega}_7 = {}^R\underset{\sim}{\omega}_1 + \hat{\underset{\sim}{\omega}}_7 \qquad \boxed{\{\omega_7\} = \{\omega_1\} + [R_1]^T\{\hat{\omega}_7\}}$$

$${}^R\underset{\sim}{\omega}_8 = {}^R\underset{\sim}{\omega}_7 + \hat{\underset{\sim}{\omega}}_8 \qquad \boxed{\{\omega_8\} = \{\omega_7\} + [R_7]^T\{\hat{\omega}_8\}}$$

b) Define the 24×1 system relative angular velocity component matrix as follows.

$$\boxed{\{\omega\}_{24\times1} = \begin{bmatrix}(\hat{\omega}_1)_1 & (\hat{\omega}_1)_2 & (\hat{\omega}_1)_3 & (\hat{\omega}_2)_1 & (\hat{\omega}_2)_2 & (\hat{\omega}_2)_3 & \cdots & (\hat{\omega}_8)_1 & (\hat{\omega}_8)_2 & (\hat{\omega}_8)_3\end{bmatrix}^T}$$

c) In the results given below, $[I]_{3\times3}$ is the 3×3 ***identity matrix***, and $[0]_{3\times3}$ is the 3×3 ***zero matrix***.

Body 1: $\boxed{\left[{}^R\omega_{1,\hat{\omega}_K}\right] = [0]_{3\times3} \quad (K\neq1)}$ $\qquad \boxed{\left[{}^R\omega_{1,\hat{\omega}_1}\right] = [I]_{3\times3}}$

Body 2: $\boxed{\left[{}^R\omega_{2,\hat{\omega}_K}\right] = \left[{}^R\omega_{1,\hat{\omega}_K}\right] \quad (K\neq2)}$ $\qquad \boxed{\left[{}^R\omega_{2,\hat{\omega}_2}\right] = [R_1]^T_{3\times3}}$

Body 3: $\boxed{\left[{}^R\omega_{3,\hat{\omega}_K}\right] = \left[{}^R\omega_{2,\hat{\omega}_K}\right] \quad (K\neq3)}$ $\qquad \boxed{\left[{}^R\omega_{3,\hat{\omega}_3}\right] = [R_2]^T_{3\times3}}$

Body 4: $\boxed{\left[{}^R\omega_{4,\hat{\omega}_K}\right] = \left[{}^R\omega_{2,\hat{\omega}_K}\right] \quad (K\neq4)}$ $\qquad \boxed{\left[{}^R\omega_{4,\hat{\omega}_4}\right] = [R_2]^T_{3\times3}}$

Body 5: $\boxed{\left[{}^R\omega_{5,\hat{\omega}_K}\right] = \left[{}^R\omega_{1,\hat{\omega}_K}\right] \quad (K\neq5)}$ $\qquad \boxed{\left[{}^R\omega_{5,\hat{\omega}_5}\right] = [R_1]^T_{3\times3}}$

Body 6: $\boxed{\left[{}^R\omega_{6,\hat{\omega}_K}\right] = \left[{}^R\omega_{1,\hat{\omega}_K}\right] \quad (K\neq6)}$ $\qquad \boxed{\left[{}^R\omega_{6,\hat{\omega}_6}\right] = [R_1]^T_{3\times3}}$

Body 7: $\boxed{\left[{}^R\omega_{7,\hat{\omega}_K}\right] = \left[{}^R\omega_{1,\hat{\omega}_K}\right] \quad (K\neq7)}$ $\qquad \boxed{\left[{}^R\omega_{7,\hat{\omega}_7}\right] = [R_1]^T_{3\times3}}$

Body 8: $\boxed{\left[{}^R\omega_{8,\hat{\omega}_K}\right] = \left[{}^R\omega_{7,\hat{\omega}_K}\right] \quad (K\neq8)}$ $\qquad \boxed{\left[{}^R\omega_{8,\hat{\omega}_8}\right] = [R_7]^T_{3\times3}}$

d) Define eight 3×24 partial angular velocity matrices $\left[{}^R\omega_{K,\omega}\right]_{3\times24}$ $(K=1,\ldots,8)$ for the system as follows.

For each body K there is a 3×24 matrix whose columns are the components of the partial angular velocity vectors associated with the elements of the angular velocity component matrix $\{\omega\}_{24\times1}$. Using the results of part (c), the partial angular velocity matrices for the system can be written as follows. The matrices $[I]$ and $[0]$ are the 3×3 ***identity*** and ***zero*** matrices, respectively.

$$\left[{}^R\omega_{K,\omega}\right]_{3\times24} \quad (K=1,\ldots,8)$$

$$\begin{array}{l} K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \end{array}\begin{bmatrix} [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [I] & [R_1]^T & [0] & [0] & [0] & [0] & [0] & [0] \\ [I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [0] \\ [I] & [R_1]^T & [0] & [R_2]^T & [0] & [0] & [0] & [0] \\ [I] & [0] & [0] & [0] & [R_1]^T & [0] & [0] & [0] \\ [I] & [0] & [0] & [0] & [0] & [R_1]^T & [0] & [0] \\ [I] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [0] \\ [I] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [R_7]^T \end{bmatrix}$$

Note that the coordinate transformation matrices are constructed using the individual relative transformation matrices. For example,

$$\boxed{[R_2]=[{}^1R_2][R_1]}$$

e) $\boxed{\{\omega_K\}_{3\times1}=\left[{}^R\omega_{K,\omega}\right]_{3\times24}\{\omega\}_{24\times1}}$ $(K=1,\ldots,8)$

Example 4

Consider again the eight-body system of Example 3. Using ***body frame components*** of the ***relative angular velocities*** of the bodies as generalized speeds, complete the following.

a) Define the ***body frame components*** of the ***angular velocities*** for all bodies in the system.

b) Combine the ***angular velocity components*** into a single 24×1 angular velocity ***system matrix*** $\{\omega'\}_{24\times1}$.

c) Define the ***body frame components*** of the ***partial angular velocities*** for all the bodies in the system.

d) Define a 3×24 ***partial angular velocity matrix*** for ***each*** body in the system.

e) Write the ***body frame components*** of the ***angular velocity*** of each body in terms of the ***system angular velocity matrix*** defined in part (b) and the ***partial angular velocity matrices*** defined in part (d).

Solution:

a) $\{\omega'_K\}$ $(K=1,\ldots,8)$ are 3×1 vectors of the ***body frame components*** of the angular velocities of the bodies.

$\{\hat{\omega}'_K\}$ $(K=1,\ldots,8)$ are 3×1 vectors of the ***body frame components*** of the angular velocities of the bodies ***relative*** to their ***base frames*** (fixed in their lower-numbered body).

${}^R\underset{\sim}{\omega}'_1 = \underset{\sim}{\hat{\omega}}'_1$	$\{\omega'_1\} = \{\hat{\omega}'_1\} = \begin{bmatrix} \hat{\omega}'_{11} & \hat{\omega}'_{12} & \hat{\omega}'_{13} \end{bmatrix}^T$
${}^R\underset{\sim}{\omega}_2 = {}^R\underset{\sim}{\omega}_1 + \underset{\sim}{\hat{\omega}}_2$	$\{\omega'_2\} = \left[{}^1R_2\right]\{\omega'_1\} + \{\hat{\omega}'_2\}$
${}^R\underset{\sim}{\omega}_3 = {}^R\underset{\sim}{\omega}_2 + \underset{\sim}{\hat{\omega}}_3$	$\{\omega'_3\} = \left[{}^2R_3\right]\{\omega'_2\} + \{\hat{\omega}'_3\}$
${}^R\underset{\sim}{\omega}_4 = {}^R\underset{\sim}{\omega}_2 + \underset{\sim}{\hat{\omega}}_4$	$\{\omega'_4\} = \left[{}^2R_4\right]\{\omega'_2\} + \{\hat{\omega}'_4\}$
${}^R\underset{\sim}{\omega}_5 = {}^R\underset{\sim}{\omega}_1 + \underset{\sim}{\hat{\omega}}_5$	$\{\omega'_5\} = \left[{}^1R_5\right]\{\omega'_1\} + \{\hat{\omega}'_5\}$
${}^R\underset{\sim}{\omega}_6 = {}^R\underset{\sim}{\omega}_1 + \underset{\sim}{\hat{\omega}}_6$	$\{\omega'_6\} = \left[{}^1R_6\right]\{\omega'_1\} + \{\hat{\omega}'_6\}$
${}^R\underset{\sim}{\omega}_7 = {}^R\underset{\sim}{\omega}_1 + \underset{\sim}{\hat{\omega}}_7$	$\{\omega'_7\} = \left[{}^1R_7\right]\{\omega'_1\} + \{\hat{\omega}'_7\}$
${}^R\underset{\sim}{\omega}_8 = {}^R\underset{\sim}{\omega}_7 + \underset{\sim}{\hat{\omega}}_8$	$\{\omega'_8\} = \left[{}^7R_8\right]\{\omega'_7\} + \{\hat{\omega}'_8\}$

b) Define the 24×1 system relative angular velocity component matrix as follows.

$$\{\omega'\}_{24\times 1} = \begin{bmatrix} (\hat{\omega}'_1)_1 & (\hat{\omega}'_1)_2 & (\hat{\omega}'_1)_3 & (\hat{\omega}'_2)_1 & (\hat{\omega}'_2)_2 & (\hat{\omega}'_2)_3 & \cdots & (\hat{\omega}'_8)_1 & (\hat{\omega}'_8)_2 & (\hat{\omega}'_8)_3 \end{bmatrix}^T$$

c) In the results given below, $[I]_{3\times 3}$ is the 3×3 identity matrix, and $[0]_{3\times 3}$ is the 3×3 zero matrix.

Body 1:	$\left[{}^R\omega'_{1,\hat{\omega}'_K}\right] = [0]_{3\times 3} \quad (K \neq 1)$	$\left[{}^R\omega'_{1,\omega'_1}\right] = [I]_{3\times 3}$
Body 2:	$\left[{}^R\omega'_{2,\omega'_K}\right] = \left[{}^1R_2\right]\left[{}^R\omega'_{1,\omega'_K}\right] \quad (K \neq 2)$	$\left[{}^R\omega'_{2,\hat{\omega}'_2}\right] = [I]_{3\times 3}$
Body 3:	$\left[{}^R\omega'_{3,\hat{\omega}'_K}\right] = \left[{}^2R_3\right]\left[{}^R\omega'_{2,\hat{\omega}'_K}\right] \quad (K \neq 3)$	$\left[{}^R\omega'_{3,\hat{\omega}'_3}\right] = [I]_{3\times 3}$
Body 4:	$\left[{}^R\omega'_{4,\hat{\omega}'_K}\right] = \left[{}^2R_4\right]\left[{}^R\omega'_{2,\hat{\omega}'_K}\right] \quad (K \neq 4)$	$\left[{}^R\omega'_{4,\hat{\omega}'_4}\right] = [I]_{3\times 3}$
Body 5:	$\left[{}^R\omega'_{5,\omega'_K}\right] = \left[{}^1R_5\right]\left[{}^R\omega'_{1,\omega'_K}\right] \quad (K \neq 5)$	$\left[{}^R\omega'_{5,\hat{\omega}'_5}\right] = [I]_{3\times 3}$
Body 6:	$\left[{}^R\omega'_{6,\omega'_K}\right] = \left[{}^1R_6\right]\left[{}^R\omega'_{1,\omega'_K}\right] \quad (K \neq 6)$	$\left[{}^R\omega'_{6,\hat{\omega}'_6}\right] = [I]_{3\times 3}$
Body 7:	$\left[{}^R\omega'_{7,\omega'_K}\right] = \left[{}^1R_7\right]\left[{}^R\omega'_{1,\omega'_K}\right] \quad (K \neq 7)$	$\left[{}^R\omega'_{7,\hat{\omega}'_7}\right] = [I]_{3\times 3}$
Body 8:	$\left[{}^R\omega'_{8,\hat{\omega}'_K}\right] = \left[{}^7R_8\right]\left[{}^R\omega'_{7,\hat{\omega}'_K}\right] \quad (K \neq 8)$	$\left[{}^R\omega'_{8,\hat{\omega}'_8}\right] = [I]_{3\times 3}$

d) Define eight 3×24 partial angular velocity matrices $\left[{}^R\omega'_{K,\omega'}\right]_{3\times 24}$ $(K=1,\ldots,8)$ for the system as follows. For each body K there is a 3×24 matrix whose columns are the components of the partial angular velocity vectors associated with the elements of the angular velocity component matrix $\{\omega'\}_{24\times 1}$. Using the results of part (c), the partial angular velocity matrices for the system can be written as follows. The matrices $[I]$ and $[0]$ are the 3×3 ***identity*** and ***zero*** matrices, respectively.

$$\left[{}^{R}\omega'_{K,\omega'}\right]_{3\times 24} \quad (K=1,\ldots,8)$$

$\searrow$

$$\begin{array}{l} K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \end{array}\begin{bmatrix} [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [{}^{1}R_2] & [I] & [0] & [0] & [0] & [0] & [0] & [0] \\ [{}^{2}R_3][{}^{1}R_2] & [{}^{2}R_3] & [I] & [0] & [0] & [0] & [0] & [0] \\ [{}^{2}R_4][{}^{1}R_2] & [{}^{2}R_4] & [0] & [I] & [0] & [0] & [0] & [0] \\ [{}^{1}R_5] & [0] & [0] & [0] & [I] & [0] & [0] & [0] \\ [{}^{1}R_6] & [0] & [0] & [0] & [0] & [I] & [0] & [0] \\ [{}^{1}R_7] & [0] & [0] & [0] & [0] & [0] & [I] & [0] \\ [{}^{7}R_8][{}^{1}R_7] & [0] & [0] & [0] & [0] & [0] & [{}^{7}R_8] & [I] \end{bmatrix}$$

Or,

$$\begin{array}{l} K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \end{array}\begin{bmatrix} [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [{}^{1}R_2] & [I] & [0] & [0] & [0] & [0] & [0] & [0] \\ [{}^{1}R_3] & [{}^{2}R_3] & [I] & [0] & [0] & [0] & [0] & [0] \\ [{}^{1}R_4] & [{}^{2}R_4] & [0] & [I] & [0] & [0] & [0] & [0] \\ [{}^{1}R_5] & [0] & [0] & [0] & [I] & [0] & [0] & [0] \\ [{}^{1}R_6] & [0] & [0] & [0] & [0] & [I] & [0] & [0] \\ [{}^{1}R_7] & [0] & [0] & [0] & [0] & [0] & [I] & [0] \\ [{}^{1}R_8] & [0] & [0] & [0] & [0] & [0] & [{}^{7}R_8] & [I] \end{bmatrix}$$

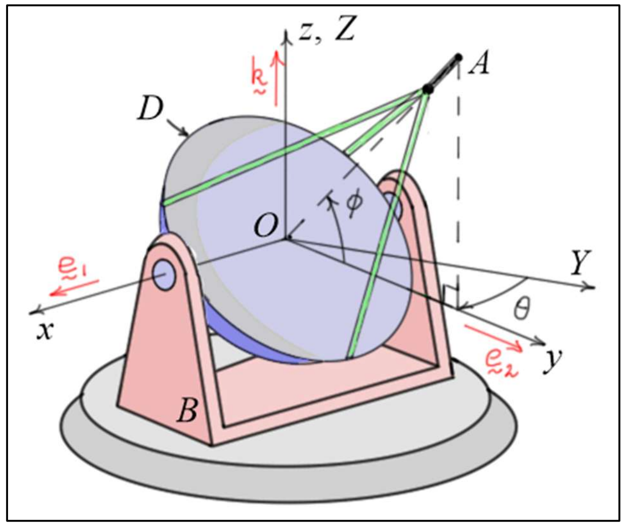
Note again that the coordinate transformation matrices are constructed using the individual relative transformation matrices. For example,

$$\boxed{[{}^{1}R_3] = [{}^{2}R_3][{}^{1}R_2]}$$

e) $$\boxed{\left\{\omega'_K\right\}_{3\times 1} = \left[{}^{R}\omega'_{K,\omega'}\right]_{3\times 24}\left\{\omega'\right\}_{24\times 1}} \quad (K=1,\ldots,8)$$

Exercises

2.1 The antenna system shown has two components, the base B and the antenna dish D. Base B rotates relative to the ground about the fixed z (or Z) axis, and dish D rotates relative to B about the rotating x-axis annotated by the unit vector $\underset{\sim}{e}_1$. At any instant, the angle between the y-axis annotated by the unit vector $\underset{\sim}{e}_2$ and the fixed Y-axis is θ, and the angle between line segment OA and the rotating y-axis is ϕ. The fixed reference frame $R:XYZ$ has its origin at O. Given the diagram, the dish is oriented relative to the fixed frame $R:XYZ$ using a 3-1 body-fixed rotation sequence with the "3" rotation about the $-Z$ axis, and the "1" rotation is about the x axis. When $\theta=\phi=0$ all reference frames align. Complete the following expressing the results in terms of the angles θ, ϕ, and their time derivatives. Use $\{\beta\}$ as the column matrix of angles θ and ϕ.



a) Find $\{\omega_D\}$ the ***fixed frame components*** and of the ***angular velocity*** of dish D in the fixed frame R and $\left[{}^R\omega_{D,\dot{\beta}}\right]$ the matrix of ***fixed frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Build the ***angular velocity vector*** as described in Examples 1 and 2.

b) Find $\{\omega'_D\}$ the ***dish-frame*** components of the ***angular velocity*** of dish D in the fixed frame R and $\left[{}^R\omega'_{D,\dot{\beta}}\right]$ the matrix of ***dish-frame*** components of the ***partial angular velocity vectors*** associated with the angle derivatives. Build the ***angular velocity vector*** as described in Examples 1 and 2.

Answers:

a) $$\{\omega_D\}=\begin{Bmatrix}0\\0\\-\dot{\theta}\end{Bmatrix}+\left[R_B\right]^T\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\\-\dot{\theta}\end{Bmatrix}+\begin{bmatrix}C_\theta & S_\theta & 0\\-S_\theta & C_\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}$$

$$\{\omega_D\}=\begin{Bmatrix}C_\theta\dot{\phi}\\-S_\theta\dot{\phi}\\-\dot{\theta}\end{Bmatrix}=\begin{bmatrix}0 & C_\theta\\0 & -S_\theta\\-1 & 0\end{bmatrix}\begin{Bmatrix}\dot{\theta}\\\dot{\phi}\end{Bmatrix}\triangleq\left[{}^R\omega_{D,\dot{\beta}}\right]\{\dot{\beta}\}$$

b) $$\{\omega'_D\}=\left[{}^BR_D\right]\begin{Bmatrix}0\\0\\-\dot{\theta}\end{Bmatrix}+\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & C_\phi & S_\phi\\0 & -S_\phi & C_\phi\end{bmatrix}\begin{Bmatrix}0\\0\\-\dot{\theta}\end{Bmatrix}+\begin{Bmatrix}\dot{\phi}\\0\\0\end{Bmatrix}$$

$$\left\{\omega'_D\right\} = \begin{Bmatrix} \dot{\phi} \\ -S_\phi \dot{\theta} \\ -C_\phi \dot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -S_\phi & 0 \\ -C_\phi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \triangleq \left[{}^R\omega'_{D,\dot{\beta}}\right]\left\{\dot{\beta}\right\}$$

2.2 Write a MATLAB script to ***numerically*** evaluate the ***matrix equations*** you derived in Exercise 2.1 using the data below. Build the ***angular velocity vectors*** first using the ***process*** used in Exercise 2.1 and then using the ***partial angular velocity matrices***.

$\theta = -30$ (deg) $\quad \phi = 60$ (deg)

$\dot{\theta} = 3(\text{rad/s}) \quad \dot{\phi} = 7(\text{rad/s})$

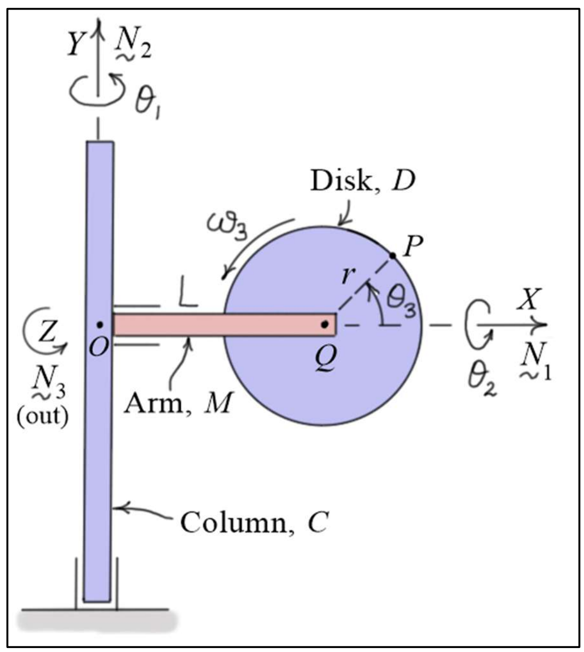
Recall that, as shown in the diagram, the angle θ is negative.

Answers:

a) $\left\{\omega_D\right\} = \begin{Bmatrix} 6.0622 \\ 3.5000 \\ -3.0000 \end{Bmatrix}$ (rad/s) $\qquad \left[{}^R\omega_{D,\dot{\beta}}\right] = \begin{bmatrix} 0 & 0.86603 \\ 0 & 0.50000 \\ -1 & 0 \end{bmatrix}$

b) $\left\{\omega'_D\right\} = \begin{Bmatrix} 7.0000 \\ -2.5981 \\ -1.5000 \end{Bmatrix}$ (rad/s) $\qquad \left[{}^R\omega'_{D,\dot{\beta}}\right] = \begin{bmatrix} 0 & 1 \\ -0.86603 & 0 \\ -0.50000 & 0 \end{bmatrix}$

2.3 The system shown has ***three*** bodies, the vertical column C, the horizontal arm M, and the disk D. Disk D has radius r and is oriented relative to M using angle θ_3. Arm M has length L and is oriented relative to C using angle θ_2. Column C is oriented relative to the ***fixed frame*** (X,Y,Z) using angle θ_1. The unit vectors $\underset{\sim}{N}_i \; (i=1,2,3)$ are along the (X,Y,Z) directions. Given the diagram, disk D is positioned relative to (X,Y,Z) using a 2-1-3 ***body fixed*** rotation sequence. Using matrix notation, complete the following. Define $\{\theta\} \triangleq \begin{bmatrix}\theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$ as the column vector of the three angles. In each case, find expressions for any general position where $\theta_1 \neq \theta_2 \neq \theta_3 \neq 0$. Note that in the position shown in the diagram, θ_1 and θ_2 are both zero. When $\theta_1 = \theta_2 = \theta_3 = 0$ all reference frames are aligned.



a) Find $\left\{\omega_D\right\}$ the ***fixed frame components*** of the ***angular velocity*** of disk D in R and $\left[{}^R\omega_{D,\dot{\theta}}\right]$ the matrix of ***fixed frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Build the ***angular velocity vector*** as described in Examples 1 and 2.

b) Find $\{\omega'_D\}$ the ***disk-frame components*** of the ***angular velocity*** of disk D in R and $\left[{}^R\omega'_{D,\dot\theta}\right]$ the matrix of ***disk-frame components*** of the ***partial angular velocity vectors*** associated with the angle derivatives. Build the ***angular velocity vector*** as described in Examples 1 and 2.

Answers:

a)

$$\{\omega_D\} = \begin{Bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{Bmatrix} + \left[R_C\right]^T \begin{Bmatrix} \dot\theta_2 \\ 0 \\ 0 \end{Bmatrix} + \left(\left[{}^C R_M\right]\left[R_C\right]\right)^T \begin{Bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{Bmatrix} + \begin{bmatrix} C_1 & 0 & S_1 \\ 0 & 1 & 0 \\ -S_1 & 0 & C_1 \end{bmatrix} \begin{Bmatrix} \dot\theta_2 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} C_1 & 0 & S_1 \\ 0 & 1 & 0 \\ -S_1 & 0 & C_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2 & -S_2 \\ 0 & S_2 & C_2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{Bmatrix}$$

$$\{\omega_D\} = \begin{Bmatrix} C_1\dot\theta_2 + S_1 C_2\dot\theta_3 \\ \dot\theta_1 - S_2\dot\theta_3 \\ -S_1\dot\theta_2 + C_1 C_2\dot\theta_3 \end{Bmatrix} = \begin{bmatrix} 0 & C_1 & S_1 C_2 \\ 1 & 0 & -S_2 \\ 0 & -S_1 & C_1 C_2 \end{bmatrix} \begin{Bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{Bmatrix} \triangleq \left[{}^R\omega_{D,\dot\theta}\right]\{\dot\theta\}$$

b)

$$\{\omega'_D\} = \left[{}^M R_D\right]\left[{}^C R_M\right] \begin{Bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{Bmatrix} + \left[{}^M R_D\right] \begin{Bmatrix} \dot\theta_2 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{Bmatrix}$$

$$= \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2 & S_2 \\ 0 & -S_2 & C_2 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{Bmatrix} + \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot\theta_2 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{Bmatrix}$$

$$\{\omega'_D\} = \begin{Bmatrix} C_2 S_3\dot\theta_1 + C_3\dot\theta_2 \\ C_2 C_3\dot\theta_1 - S_3\dot\theta_2 \\ -S_2\dot\theta_1 + \dot\theta_3 \end{Bmatrix} = \begin{bmatrix} C_2 S_3 & C_3 & 0 \\ C_2 C_3 & -S_3 & 0 \\ -S_2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{Bmatrix} \triangleq \left[{}^R\omega'_{D,\dot\theta}\right]\{\dot\theta\}$$

2.4 Write a MATLAB script to ***numerically evaluate*** the equations you derived in Exercise 2.3 using the data below. Build the ***angular velocity vectors*** first using the ***process*** used in Exercise 2.1 and then using the ***partial angular velocity matrices***.

$\theta_1 = 20$ (deg) $\quad\theta_2 = 40$ (deg) $\quad\theta_3 = 60$ (deg)

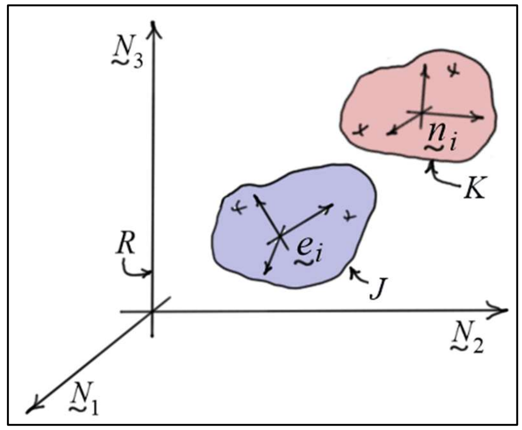
$\dot\theta_1 = 2$ (rad/s) $\quad\dot\theta_2 = -3$ (rad/s) $\quad\dot\theta_3 = 5$ (rad/s)

Answers:

a) $\{\omega_D\} = \begin{Bmatrix} -1.5091 \\ -1.2139 \\ 4.6253 \end{Bmatrix}$ (rad/s) $\qquad \left[{}^R\omega_{D,\dot\theta}\right] = \begin{bmatrix} 0 & 0.93969 & 0.26200 \\ 1 & 0 & -0.64279 \\ 0 & -0.34202 & 0.71985 \end{bmatrix}$

b) $\left\{\omega'_D\right\} = \begin{Bmatrix} -0.17317 \\ 3.3641 \\ 3.7144 \end{Bmatrix}$ (rad/s) $\left[{}^R\omega'_{D,\dot{\theta}}\right] = \begin{bmatrix} 0.66341 & 0.50000 & 0 \\ 0.38302 & -0.86603 & 0 \\ -0.64279 & 0 & 1 \end{bmatrix}$

2.5 The two bodies shown are part of a multibody system. Body J is ***oriented*** with respect to the ***fixed frame*** R and body K is ***oriented*** with respect to ***body*** J both using 2-3-1 body fixed rotation sequences. The angles $\theta_{Ji}\ (i=1,2,3)$ give the orientation of ***body*** J ***relative*** to the ***fixed frame*** R, and the angles $\hat{\theta}_{Ki}\ (i=1,2,3)$ give the orientation of ***body*** K ***relative*** to ***body*** J. Use the matrices $\{\theta_J\}_{3\times1}$ and $\{\theta_K\}_{3\times1}$ as the column matrices of angles $\theta_{Ji}\ (i=1,2,3)$ and $\hat{\theta}_{Ki}\ (i=1,2,3)$, respectively. Complete the following in terms of the angles $\theta_{Ji}\ (i=1,2,3)$, $\hat{\theta}_{Ki}\ (i=1,2,3)$, and their time derivatives.



a) Find $\{\omega_J\}$ the ***base frame components*** of the ***angular velocity*** of body J relative to its ***base frame*** R, and find $\left[{}^R\omega_{J,\dot{\theta}_J}\right]$ and $\left[{}^R\omega_{J,\dot{\theta}_K}\right]$ the matrices of ***base frame components*** of the ***partial angular velocity vectors*** of body J associated with the ***angle derivatives***. Note that the ***base frame*** of ***body*** J is the ***fixed frame***. ***Build*** the ***angular velocity vector*** as described in Examples 1 and 2. Express the results in matrix form.

b) Find $\left\{{}^J\omega_K\right\}$ the ***base frame components*** of the ***angular velocity*** of body K ***relative*** to its base frame body J, and find $\left[{}^J\omega_{K,\dot{\theta}_J}\right]$ and $\left[{}^J\omega_{K,\dot{\theta}_K}\right]$ the matrices of ***base frame components*** of the ***partial angular velocity vectors*** of body K with respect to its base frame associated with the ***angle derivatives***. ***Build*** the ***angular velocity vector*** as described in Examples 1 and 2. Express the results in matrix form.

c) Find $\{\omega_K\}$ the ***fixed frame components*** of the ***angular velocity*** of body K relative to the fixed frame R, and find $\left[{}^R\omega_{K,\dot{\theta}_J}\right]$ and $\left[{}^R\omega_{K,\dot{\theta}_K}\right]$ the matrices of ***fixed frame components*** of the ***partial angular velocity vectors*** of body K associated with the ***angle derivatives***. ***Build*** the ***angular velocity vector*** using the ***summation rule*** for angular velocities and the results from parts (a) and (b).

d) Find $\{\omega'_J\}$ the ***body frame components*** of the ***angular velocity*** of body J ***relative*** to its ***base frame*** R, and find $\left[{}^R\omega'_{J,\dot{\theta}_J}\right]$ and $\left[{}^R\omega'_{J,\dot{\theta}_K}\right]$ the matrices of ***body-frame components*** of the ***partial angular velocity vectors*** of body J associated with the ***angle derivatives***. ***Build*** the ***angular velocity vector*** as described in Examples 1 and 2. Express the results in matrix form.

e) Find $\left\{{}^{J}\omega'_K\right\}$ the ***body frame components*** of the ***angular velocity*** of body K relative to its base frame (body J), and find $\left[{}^{J}\omega'_{K,\dot{\theta}_J}\right]$ and $\left[{}^{J}\omega'_{K,\dot{\theta}_K}\right]$ the matrices of ***body-frame components*** of the ***partial angular velocity vectors*** of body K associated with the ***angle derivatives***. ***Build*** the ***angular velocity vector*** as described in Examples 1 and 2. Express the results in matrix form.

f) Find $\left\{\omega'_K\right\}$ the ***body frame components*** of the ***angular velocity*** of body K relative to the fixed frame R, and find $\left[{}^{R}\omega'_{K,\dot{\theta}_J}\right]$ and $\left[{}^{R}\omega'_{K,\dot{\theta}_K}\right]$ the matrices of ***body frame components*** of the ***partial angular velocity vectors*** of body K associated with the ***angle derivatives***. Build the ***angular velocity vector*** using the ***summation rule*** for angular velocities and the results from parts (d) and (e).

Answers:

Note that reference frames JR' and JR'' are the ***intermediate reference frames*** used to orient body J relative to the ground, and reference frames KR' and KR'' are the ***intermediate reference frames*** used to orient body K relative to body J.

a)

$$\left\{\omega_J\right\}=\begin{Bmatrix}0\\\dot{\theta}_{J1}\\0\end{Bmatrix}+\left[{}^{R}R_{JR'}\right]^T\begin{Bmatrix}0\\0\\\dot{\theta}_{J2}\end{Bmatrix}+\left(\left[{}^{JR'}R_{JR''}\right]\left[{}^{R}R_{JR'}\right]\right)^T\begin{Bmatrix}\dot{\theta}_{J3}\\0\\0\end{Bmatrix}$$

$$=\begin{Bmatrix}0\\\dot{\theta}_{J1}\\0\end{Bmatrix}+\begin{bmatrix}C_{J1}&0&S_{J1}\\0&1&0\\-S_{J1}&0&C_{J1}\end{bmatrix}\begin{Bmatrix}0\\0\\\dot{\theta}_{J2}\end{Bmatrix}+\begin{bmatrix}C_{J1}&0&S_{J1}\\0&1&0\\-S_{J1}&0&C_{J1}\end{bmatrix}\begin{bmatrix}C_{J2}&-S_{J2}&0\\S_{J2}&C_{J2}&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{J3}\\0\\0\end{Bmatrix}$$

$$\left\{\omega_J\right\}=\begin{bmatrix}0&S_{J1}&C_{J1}C_{J2}\\1&0&S_{J2}\\0&C_{J1}&-S_{J1}C_{J2}\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{J1}\\\dot{\theta}_{J2}\\\dot{\theta}_{J3}\end{Bmatrix}\triangleq\left[{}^{R}\omega_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\}+\underbrace{\left[{}^{R}\omega_{J,\dot{\theta}_K}\right]}_{\text{zero}}\left\{\dot{\theta}_K\right\}$$

b)

$$\left\{{}^{J}\omega_K\right\}\triangleq\left\{\hat{\omega}_K\right\}=\begin{Bmatrix}0\\\dot{\hat{\theta}}_{K1}\\0\end{Bmatrix}+\left[{}^{J}R_{KR'}\right]^T\begin{Bmatrix}0\\0\\\dot{\hat{\theta}}_{K2}\end{Bmatrix}+\left(\left[{}^{KR'}R_{KR''}\right]\left[{}^{J}R_{KR'}\right]\right)^T\begin{Bmatrix}\dot{\hat{\theta}}_{K3}\\0\\0\end{Bmatrix}$$

$$=\begin{Bmatrix}0\\\dot{\hat{\theta}}_{K1}\\0\end{Bmatrix}+\begin{bmatrix}C_{K1}&0&-S_{K1}\\0&1&0\\S_{K1}&0&C_{K1}\end{bmatrix}^T\begin{Bmatrix}0\\0\\\dot{\hat{\theta}}_{K2}\end{Bmatrix}+\begin{bmatrix}C_{K1}&0&-S_{K1}\\0&1&0\\S_{K1}&0&C_{K1}\end{bmatrix}^T\begin{bmatrix}C_{K2}&S_{K2}&0\\-S_{K2}&C_{K2}&0\\0&0&1\end{bmatrix}^T\begin{Bmatrix}\dot{\hat{\theta}}_{K3}\\0\\0\end{Bmatrix}$$

$$\left\{\hat{\omega}_K\right\}=\begin{bmatrix}0&S_{K1}&C_{K1}C_{K2}\\1&0&S_{K2}\\0&C_{K1}&-S_{K1}C_{K2}\end{bmatrix}\begin{Bmatrix}\dot{\hat{\theta}}_{K1}\\\dot{\hat{\theta}}_{K2}\\\dot{\hat{\theta}}_{K3}\end{Bmatrix}\triangleq\underbrace{\left[{}^{J}\omega_{K,\dot{\theta}_J}\right]}_{\text{zero}}\left\{\dot{\theta}_J\right\}+\left[{}^{J}\omega_{K,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$

c) $\left\{\omega_K\right\}=\left\{\omega_J\right\}+\left[R_J\right]^T\left\{{}^J\omega_K\right\}=\left\{\omega_J\right\}+\left[R_J\right]^T\left\{\hat{\omega}_K\right\}$

$$\left\{\omega_K\right\}=\left\{\omega_J\right\}+\left[R_J\right]^T\left\{\hat{\omega}_K\right\}=\left[{}^R\omega_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\}+\left[R_J\right]^T\left[{}^J\omega_{K,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$
$$\triangleq\left[{}^R\omega_{K,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\}+\left[{}^R\omega_{K,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$

$$\left[{}^R\omega_{K,\dot{\theta}_J}\right]=\left[{}^R\omega_{J,\dot{\theta}_J}\right]=\begin{bmatrix}0 & S_{J1} & C_{J1}C_{J2}\\ 1 & 0 & S_{J2}\\ 0 & C_{J1} & -S_{J1}C_{J2}\end{bmatrix}$$

$$\left[{}^R\omega_{K,\dot{\theta}_K}\right]=\left[R_J\right]^T\left[{}^J\omega_{K,\dot{\theta}_K}\right]=\left[R_J\right]^T\begin{bmatrix}0 & S_{K1} & C_{K1}C_{K2}\\ 1 & 0 & S_{K2}\\ 0 & C_{K1} & -S_{K1}C_{K2}\end{bmatrix}$$

d)

$$\left\{\omega'_J\right\}=\left[{}^{JR''}R_J\right]\left[{}^{JR'}R_{JR''}\right]\begin{Bmatrix}0\\ \dot{\theta}_{J1}\\ 0\end{Bmatrix}+\left[{}^{JR''}R_J\right]\begin{Bmatrix}0\\ 0\\ \dot{\theta}_{J2}\end{Bmatrix}+\begin{Bmatrix}\dot{\theta}_{J3}\\ 0\\ 0\end{Bmatrix}$$
$$=\begin{bmatrix}1 & 0 & 0\\ 0 & C_{J3} & S_{J3}\\ 0 & -S_{J3} & C_{J3}\end{bmatrix}\begin{bmatrix}C_{J2} & S_{J2} & 0\\ -S_{J2} & C_{J2} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}0\\ \dot{\theta}_{J1}\\ 0\end{Bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & C_{J3} & S_{J3}\\ 0 & -S_{J3} & C_{J3}\end{bmatrix}\begin{Bmatrix}0\\ 0\\ \dot{\theta}_{J2}\end{Bmatrix}+\begin{Bmatrix}\dot{\theta}_{J3}\\ 0\\ 0\end{Bmatrix}$$

$$\left\{\omega'_J\right\}=\begin{bmatrix}S_{J2} & 0 & 1\\ C_{J2}C_{J3} & S_{J3} & 0\\ -C_{J2}S_{J3} & C_{J3} & 0\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{J1}\\ \dot{\theta}_{J2}\\ \dot{\theta}_{J3}\end{Bmatrix}\triangleq\left[{}^R\omega'_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\}+\underbrace{\left[{}^R\omega'_{J,\dot{\theta}_K}\right]}_{\text{zero}}\left\{\dot{\theta}_K\right\}$$

e)

$$\left\{{}^J\omega'_K\right\}\triangleq\left\{\hat{\omega}'_K\right\}=\left[{}^{KR''}R_K\right]\left[{}^{KR'}R_{KR''}\right]\begin{Bmatrix}0\\ \dot{\hat{\theta}}_{K1}\\ 0\end{Bmatrix}+\left[{}^{KR''}R_K\right]\begin{Bmatrix}0\\ 0\\ \dot{\hat{\theta}}_{K2}\end{Bmatrix}+\begin{Bmatrix}\dot{\hat{\theta}}_{K3}\\ 0\\ 0\end{Bmatrix}$$
$$=\begin{bmatrix}1 & 0 & 0\\ 0 & C_{K3} & S_{K3}\\ 0 & -S_{K3} & C_{K3}\end{bmatrix}\begin{bmatrix}C_{K2} & S_{K2} & 0\\ -S_{K2} & C_{K2} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}0\\ \dot{\hat{\theta}}_{K1}\\ 0\end{Bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & C_{K3} & S_{K3}\\ 0 & -S_{K3} & C_{K3}\end{bmatrix}\begin{Bmatrix}0\\ 0\\ \dot{\hat{\theta}}_{K2}\end{Bmatrix}+\begin{Bmatrix}\dot{\hat{\theta}}_{K3}\\ 0\\ 0\end{Bmatrix}$$

$$\left\{\hat{\omega}'_K\right\}=\begin{bmatrix}S_{K2} & 0 & 1\\ C_{K2}C_{K3} & S_{K3} & 0\\ -C_{K2}S_{K3} & C_{K3} & 0\end{bmatrix}\begin{Bmatrix}\dot{\hat{\theta}}_{K1}\\ \dot{\hat{\theta}}_{K2}\\ \dot{\hat{\theta}}_{K3}\end{Bmatrix}=\underbrace{\left[{}^J\omega'_{K,\dot{\theta}_J}\right]}_{\text{zero}}\left\{\dot{\theta}_J\right\}+\left[{}^J\omega'_{K,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$

f)
$$\left\{\omega'_K\right\} = \left[{}^J R_K\right]\left\{\omega'_J\right\} + \left\{\hat{\omega}'_K\right\} = \left[{}^J R_K\right]\left[{}^R\omega'_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^J\omega'_{K,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$
$$\triangleq \left[{}^R\omega'_{K,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} + \left[{}^R\omega'_{K,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$

$$\left[{}^R\omega'_{K,\dot{\theta}_J}\right] = \left[{}^J R_K\right]\left[{}^R\omega'_{J,\dot{\theta}_J}\right] = \left[{}^J R_K\right]\begin{bmatrix} S_{J2} & 0 & 1 \\ C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \end{bmatrix}$$

$$\left[{}^R\omega'_{K,\dot{\theta}_K}\right] = \left[{}^J\omega'_{K,\dot{\theta}_K}\right] = \begin{bmatrix} S_{K2} & 0 & 1 \\ C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \end{bmatrix}$$

2.6 Write a MATLAB script to ***numerically evaluate*** the equations you derived in Exercise 2.5 using the data below. Build the ***angular velocity vectors*** first using the ***process*** used in Examples 1 and 2 and then using the ***partial angular velocity matrices***.

$\theta_{J1} = 20$ (deg) $\quad \theta_{J2} = 40$ (deg) $\quad \theta_{J3} = 60$ (deg)

$\dot{\theta}_{J1} = 2$ (rad/s) $\quad \dot{\theta}_{J2} = -3$ (rad/s) $\quad \dot{\theta}_{J3} = 5$ (rad/s)

$\hat{\theta}_{K1} = -30$ (deg) $\quad \hat{\theta}_{K2} = -20$ (deg) $\quad \hat{\theta}_{K3} = 40$ (deg)

$\dot{\hat{\theta}}_{K1} = -5$ (rad/s) $\quad \dot{\hat{\theta}}_{K2} = 4$ (rad/s) $\quad \dot{\hat{\theta}}_{K3} = 3$ (rad/s)

Answers:

a) $\left\{\omega_J\right\} = \begin{Bmatrix} 2.5732 \\ 5.2139 \\ -4.1291 \end{Bmatrix}$ (rad/s) $\quad \left[{}^R\omega_{J,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & 0.34202 & 0.71985 \\ 1 & 0 & 0.64279 \\ 0 & 0.93969 & -0.26200 \end{bmatrix}$ $\quad \left[{}^R\omega_{J,\dot{\theta}_K}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $\left\{\hat{\omega}_K\right\} \triangleq \left\{{}^J\omega_K\right\} = \begin{Bmatrix} 0.44139 \\ -6.0261 \\ 4.8736 \end{Bmatrix}$ (rad/s) $\quad \left[{}^J\omega_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \left[{}^J\omega_{K,\dot{\theta}_K}\right] = \begin{bmatrix} 0 & -0.50000 & 0.81380 \\ 1 & 0 & -0.34202 \\ 0 & 0.86603 & 0.46985 \end{bmatrix}$

c) $\left\{\omega_K\right\} = \begin{Bmatrix} 6.3088 \\ -0.043696 \\ -8.4492 \end{Bmatrix}$ (rad/s) $\quad \left[{}^R\omega_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & 0.34202 & 0.71985 \\ 1 & 0 & 0.64279 \\ 0 & 0.93969 & -0.26200 \end{bmatrix}$

$$\left[{}^R\omega_{K,\dot{\theta}_K}\right] = \begin{bmatrix} -0.0058133 & 0.24119 & 0.91392 \\ 0.38302 & -0.89593 & 0.080395 \\ 0.92372 & 0.37302 & -0.39785 \end{bmatrix}$$

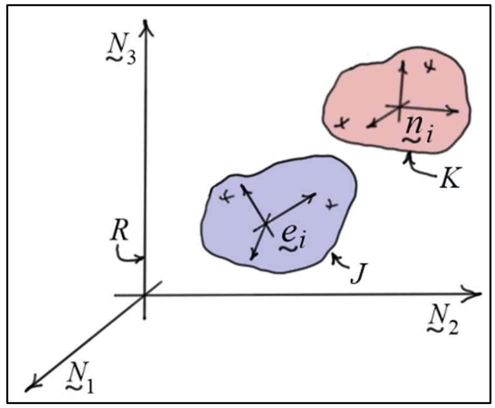
d) $\left\{\omega'_J\right\} = \begin{Bmatrix} 6.2856 \\ -1.8320 \\ -2.8268 \end{Bmatrix}$ (rad/s) $\quad \left[{}^R\omega'_{J,\dot{\theta}_J}\right] = \begin{bmatrix} 0.64279 & 0 & 1 \\ 0.38302 & 0.86603 & 0 \\ -0.66341 & 0.50000 & 0 \end{bmatrix}$ $\quad \left[{}^R\omega'_{J,\dot{\theta}_K}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

e) $\left\{\hat{\omega}'_K\right\} \triangleq \left\{{}^J\omega'_K\right\} = \begin{Bmatrix} 4.7101 \\ -1.0281 \\ 6.0843 \end{Bmatrix}$ (rad/s) $\left[{}^J\omega'_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\left[{}^J\omega'_{K,\dot{\theta}_K}\right] = \begin{bmatrix} -0.34202 & 0 & 1 \\ 0.71985 & 0.64279 & 0 \\ -0.60402 & 0.76604 & 0 \end{bmatrix}$

f) $\left\{\omega'_K\right\} = \begin{Bmatrix} 9.1237 \\ -4.8847 \\ 2.0220 \end{Bmatrix}$ (rad/s) $\left[{}^R\omega'_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 0.080395 & -0.061275 & 0.81380 \\ -0.24123 & 0.96724 & -0.094493 \\ -0.96713 & -0.24635 & -0.57341 \end{bmatrix}$

$\left[{}^R\omega'_{K,\dot{\theta}_K}\right] = \begin{bmatrix} -0.34202 & 0 & 1 \\ 0.71985 & 0.64279 & 0 \\ -0.60402 & 0.76604 & 0 \end{bmatrix}$

2.7 Consider again the two body system of Exercise 2.5. As before, ***body*** J is ***oriented*** with respect to the ***fixed frame*** R and ***body*** K is ***oriented*** with respect to ***body*** J both using body fixed orientation angle sequences. The ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_J$ the angular velocity of body J relative to the fixed frame are $\omega_{Ji}\,(i=1,2,3)$ and $\omega'_{Ji}\,(i=1,2,3)$, respectively. The ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body K relative to the fixed frame are $\omega_{Ki}\,(i=1,2,3)$ and $\omega'_{Ki}\,(i=1,2,3)$, respectively. The ***base frame*** (body J frame) and ***body frame*** components of ${}^J\underset{\sim}{\omega}_K$ the angular velocity of ***body*** K relative to ***body*** J are $\hat{\omega}_{Ki}\,(i=1,2,3)$ and $\hat{\omega}'_{Ki}\,(i=1,2,3)$, respectively. Complete the following.



a) Find $\left[{}^R\omega_{J,\omega_J}\right]$, $\left[{}^R\omega_{J,\hat{\omega}_K}\right]$, $\left[{}^J\omega_{K,\omega_J}\right]$ and $\left[{}^J\omega_{K,\hat{\omega}_K}\right]$ the matrices of ***base frame*** components of the ***partial angular velocity*** vectors of the bodies associated with the ***angular velocity components*** $\omega_{Ji}\,(i=1,2,3)$ and $\hat{\omega}_{Ki}\,(i=1,2,3)$. As before, the base frame of body J is the fixed frame, and the base frame of body K is the body J frame.

b) Find $\left\{\omega_K\right\}$ the ***fixed frame components*** of the ***angular velocity*** of body K relative to the fixed frame R, and find $\left[{}^R\omega_{K,\omega_J}\right]$ and $\left[{}^R\omega_{K,\hat{\omega}_K}\right]$ the matrices of ***fixed frame components*** of the ***partial angular velocity vectors*** of body K associated with the ***angular velocity components*** $\omega_{Ji}\,(i=1,2,3)$ and $\hat{\omega}_{Ki}\,(i=1,2,3)$.

c) Find $\left[{}^R\omega'_{J,\omega'_J}\right]$, $\left[{}^R\omega'_{J,\hat{\omega}'_K}\right]$, $\left[{}^J\omega'_{K,\omega'_J}\right]$ and $\left[{}^J\omega'_{K,\hat{\omega}'_K}\right]$ the matrices of ***body frame*** components of the ***partial angular velocity*** vectors of the bodies associated with the ***angular velocity components*** $\omega'_{Ji}\,(i=1,2,3)$ and $\hat{\omega}'_{Ki}\,(i=1,2,3)$.

d) Find $\{\omega'_K\}$ the ***body frame components*** of the ***angular velocity*** of body K relative to the fixed frame R, and find $\left[{}^R\omega'_{K,\omega'_J}\right]$ and $\left[{}^R\omega'_{K,\hat{\omega}'_K}\right]$ the matrices of ***body frame components*** of the ***partial angular velocity vectors*** of body K associated with the ***angular velocity components*** $\omega'_{Ji}\ (i=1,2,3)$ and $\hat{\omega}'_{Ki}\ (i=1,2,3)$.

Answers:

a)
$$\{\omega_J\} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix} \triangleq \left[{}^R\omega_{J,\omega_J}\right]\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix} + \underbrace{\left[{}^R\omega_{J,\hat{\omega}_K}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}$$

$$\{{}^J\omega_K\} \triangleq \{\hat{\omega}_K\} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix} \triangleq \underbrace{\left[{}^J\omega_{K,\omega_J}\right]}_{\text{zero}}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix} + \left[{}^J\omega_{K,\hat{\omega}_K}\right]\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}$$

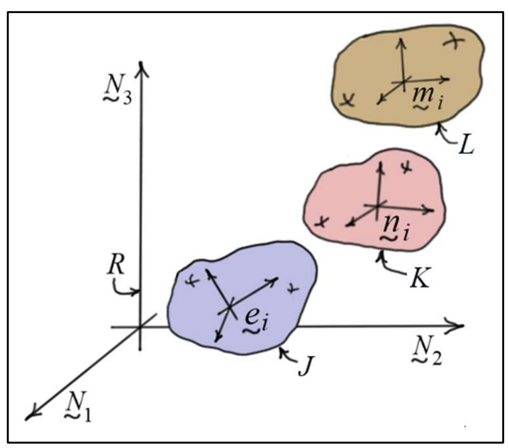
b)
$$\{\omega_K\} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\{\omega_J\} + \left[R_J\right]^T\{\hat{\omega}_K\} \triangleq \left[{}^R\omega_{K,\omega_J}\right]\{\omega_J\} + \left[{}^R\omega_{K,\hat{\omega}_K}\right]\{\hat{\omega}_K\}$$

c)
$$\{\omega'_J\} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega'_{J1}\\\omega'_{J2}\\\omega'_{J3}\end{Bmatrix} \triangleq \left[{}^R\omega'_{J,\omega'_J}\right]\begin{Bmatrix}\omega'_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix} + \underbrace{\left[{}^R\omega'_{J,\hat{\omega}'_K}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}'_{K1}\\\hat{\omega}'_{K2}\\\hat{\omega}'_{K3}\end{Bmatrix}$$

$$\{{}^J\omega'_K\} \triangleq \{\hat{\omega}'_K\} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\hat{\omega}'_{K1}\\\hat{\omega}'_{K2}\\\hat{\omega}'_{K3}\end{Bmatrix} \triangleq \underbrace{\left[{}^J\omega'_{K,\omega'_J}\right]}_{\text{zero}}\begin{Bmatrix}\omega'_{J1}\\\omega'_{J2}\\\omega'_{J3}\end{Bmatrix} + \left[{}^J\omega'_{K,\hat{\omega}'_K}\right]\begin{Bmatrix}\hat{\omega}'_{K1}\\\hat{\omega}'_{K2}\\\hat{\omega}'_{K3}\end{Bmatrix}$$

d)
$$\{\omega'_K\} = \left[{}^JR_K\right]\{\omega'_J\} + \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\{\hat{\omega}'_K\} \triangleq \left[{}^R\omega'_{K,\omega'_J}\right]\{\omega'_J\} + \left[{}^R\omega'_{K,\hat{\omega}'_K}\right]\{\hat{\omega}'_K\}$$

2.8 Consider now a system with three bodies. As before, body J is ***oriented*** with respect to the ***fixed frame*** R. Body K is ***oriented*** with respect to ***body*** J, and body L is ***oriented*** with respect to ***body*** K. Body fixed orientation angle sequences are used to describe the orientation of all three bodies relative to their base frames. The ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_B\ (B=J,K,L)$ the angular velocities of bodies relative to the fixed frame are $\omega_{Bi}\ (B=J,K,L;\ i=1,2,3)$ and $\omega'_{Bi}\ (B=J,K,L;\ i=1,2,3)$,



respectively. The ***base frame*** and ***body frame*** components of ${}^{J}\underset{\sim}{\omega}_K$ and ${}^{K}\underset{\sim}{\omega}_L$ the angular velocity of bodies K and L relative to their adjacent bodies are $\hat{\omega}_{Bi}\left(B=K,L;\ i=1,2,3\right)$ and $\hat{\omega}'_{Bi}\left(B=K,L;\ i=1,2,3\right)$, respectively. Complete the following.

a) Find $\left[{}^{R}\omega_{J,\omega_J}\right]$, $\left[{}^{R}\omega_{J,\hat{\omega}_K}\right]$, $\left[{}^{R}\omega_{J,\hat{\omega}_L}\right]$, $\left[{}^{J}\omega_{K,\omega_J}\right]$, $\left[{}^{J}\omega_{K,\hat{\omega}_K}\right]$, $\left[{}^{J}\omega_{K,\hat{\omega}_L}\right]$, $\left[{}^{K}\omega_{L,\omega_J}\right]$, $\left[{}^{K}\omega_{L,\hat{\omega}_K}\right]$, and $\left[{}^{K}\omega_{L,\hat{\omega}_L}\right]$ the matrices of ***base frame*** components of the ***partial angular velocity*** vectors of the bodies associated with the ***angular velocity components*** $\omega_{Ji}\left(i=1,2,3\right)$, $\hat{\omega}_{Ki}\left(i=1,2,3\right)$, and $\hat{\omega}_{Li}\left(i=1,2,3\right)$. The base frame of body J is the fixed frame, the base frame of body K is the body J frame, and the base frame of body L is the body K frame.

b) Find $\left\{\omega_K\right\}$ and $\left\{\omega_L\right\}$ the ***fixed frame components*** of the ***angular velocities*** of bodies K and L relative to the fixed frame R, and find $\left[{}^{R}\omega_{K,\omega_J}\right]$, $\left[{}^{R}\omega_{K,\hat{\omega}_K}\right]$, $\left[{}^{R}\omega_{K,\hat{\omega}_L}\right]$, $\left[{}^{R}\omega_{L,\omega_J}\right]$, $\left[{}^{R}\omega_{L,\hat{\omega}_K}\right]$, and $\left[{}^{R}\omega_{L,\hat{\omega}_L}\right]$ the matrices of ***fixed frame components*** of the ***partial angular velocity vectors*** of bodies K and L associated with the ***angular velocity components*** $\omega_{Ji}\left(i=1,2,3\right)$, $\hat{\omega}_{Ki}\left(i=1,2,3\right)$, and $\hat{\omega}_{Li}\left(i=1,2,3\right)$.

c) Find $\left[{}^{R}\omega'_{J,\omega'_J}\right]$, $\left[{}^{R}\omega'_{J,\hat{\omega}'_K}\right]$, $\left[{}^{R}\omega'_{J,\hat{\omega}'_L}\right]$, $\left[{}^{J}\omega'_{K,\omega'_J}\right]$, $\left[{}^{J}\omega'_{K,\hat{\omega}'_K}\right]$,$\left[{}^{J}\omega'_{K,\hat{\omega}'_L}\right]$, $\left[{}^{K}\omega'_{L,\omega'_J}\right]$, $\left[{}^{K}\omega'_{L,\hat{\omega}'_K}\right]$, $\left[{}^{K}\omega'_{L,\hat{\omega}'_L}\right]$ the matrices of ***body frame*** components of the ***partial angular velocity*** vectors of the bodies associated with the ***angular velocity components*** $\omega'_{Ji}\left(i=1,2,3\right)$, $\hat{\omega}'_{Ki}\left(i=1,2,3\right)$, and $\hat{\omega}'_{Li}\left(i=1,2,3\right)$.

d) Find $\left\{\omega'_K\right\}$ and $\left\{\omega'_L\right\}$ the ***body frame components*** of the ***angular velocities*** of bodies K and L relative to the fixed frame R, and find $\left[{}^{R}\omega'_{K,\omega'_J}\right]$, $\left[{}^{R}\omega'_{K,\hat{\omega}'_K}\right]$, $\left[{}^{R}\omega'_{K,\hat{\omega}'_L}\right]$, $\left[{}^{R}\omega'_{L,\omega'_J}\right]$, $\left[{}^{R}\omega'_{L,\hat{\omega}'_K}\right]$, $\left[{}^{R}\omega'_{L,\hat{\omega}'_L}\right]$ the matrices of ***body frame components*** of the ***partial angular velocity vectors*** of bodies K and L associated with the ***angular velocity components*** $\omega'_{Ji}\left(i=1,2,3\right)$, $\hat{\omega}'_{Ki}\left(i=1,2,3\right)$, and $\hat{\omega}'_{Li}\left(i=1,2,3\right)$.

Answers:

a)
$$\left\{\omega_J\right\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}\triangleq\left[{}^{R}\omega_{J,\omega_J}\right]\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}+\underbrace{\left[{}^{R}\omega_{J,\hat{\omega}_K}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}+\underbrace{\left[{}^{R}\omega_{J,\hat{\omega}_L}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{L1}\\\hat{\omega}_{L2}\\\hat{\omega}_{L3}\end{Bmatrix}$$

$$\left\{{}^{J}\omega_K\right\}\triangleq\left\{\hat{\omega}_K\right\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}\triangleq\underbrace{\left[{}^{J}\omega_{K,\omega_J}\right]}_{\text{zero}}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}+\left[{}^{J}\omega_{K,\hat{\omega}_K}\right]\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}+\underbrace{\left[{}^{J}\omega_{K,\hat{\omega}_L}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{L1}\\\hat{\omega}_{L2}\\\hat{\omega}_{L3}\end{Bmatrix}$$

$$\left\{{}^K\omega_L\right\} \triangleq \left\{\hat{\omega}_L\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\omega}_{L1} \\ \hat{\omega}_{L2} \\ \hat{\omega}_{L3} \end{Bmatrix} \triangleq \underbrace{\left[{}^J\omega_{K,\omega_J}\right]}_{\text{zero}} \begin{Bmatrix} \omega_{J1} \\ \omega_{J2} \\ \omega_{J3} \end{Bmatrix} + \underbrace{\left[{}^J\omega_{K,\hat{\omega}_K}\right]}_{\text{zero}} \begin{Bmatrix} \hat{\omega}_{K1} \\ \hat{\omega}_{K2} \\ \hat{\omega}_{K3} \end{Bmatrix} + \left[{}^J\omega_{K,\hat{\omega}_L}\right] \begin{Bmatrix} \hat{\omega}_{L1} \\ \hat{\omega}_{L2} \\ \hat{\omega}_{L3} \end{Bmatrix}$$

b) $$\left\{\omega_K\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\omega_J\right\} + \left[R_J\right]^T \left\{\hat{\omega}_K\right\} \triangleq \left[{}^R\omega_{K,\omega_J}\right]\left\{\omega_J\right\} + \left[{}^R\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\} + \underbrace{\left[{}^R\omega_{K,\hat{\omega}_L}\right]}_{\text{zero}}\left\{\hat{\omega}_L\right\}$$

$$\left\{\omega_L\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\omega_J\right\} + \left[R_J\right]^T \left\{\hat{\omega}_K\right\} + \left[R_K\right]^T \left\{\hat{\omega}_L\right\}$$
$$\triangleq \left[{}^R\omega_{K,\omega_J}\right]\left\{\omega_J\right\} + \left[{}^R\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\} + \left[{}^R\omega_{K,\hat{\omega}_L}\right]\left\{\hat{\omega}_L\right\}$$

c) $$\left\{\omega'_J\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} \triangleq \left[{}^R\omega'_{J,\omega'_J}\right] \begin{Bmatrix} \omega'_{J1} \\ \omega_{J2} \\ \omega_{J3} \end{Bmatrix} + \underbrace{\left[{}^R\omega'_{J,\hat{\omega}'_K}\right]}_{\text{zero}} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} + \underbrace{\left[{}^R\omega'_{J,\hat{\omega}'_L}\right]}_{\text{zero}} \begin{Bmatrix} \hat{\omega}'_{L1} \\ \hat{\omega}'_{L2} \\ \hat{\omega}'_{L3} \end{Bmatrix}$$

$$\left\{{}^J\omega'_K\right\} \triangleq \left\{\hat{\omega}'_K\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} \triangleq \underbrace{\left[{}^J\omega'_{K,\omega'_J}\right]}_{\text{zero}} \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} + \left[{}^J\omega'_{K,\hat{\omega}'_K}\right] \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} + \underbrace{\left[{}^J\omega'_{K,\hat{\omega}'_L}\right]}_{\text{zero}} \begin{Bmatrix} \hat{\omega}'_{L1} \\ \hat{\omega}'_{L2} \\ \hat{\omega}'_{L3} \end{Bmatrix}$$

$$\left\{{}^K\omega'_L\right\} \triangleq \left\{\hat{\omega}'_L\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\omega}'_{L1} \\ \hat{\omega}'_{L2} \\ \hat{\omega}'_{L3} \end{Bmatrix} \triangleq \underbrace{\left[{}^J\omega'_{K,\omega'_J}\right]}_{\text{zero}} \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} + \underbrace{\left[{}^J\omega'_{K,\hat{\omega}'_K}\right]}_{\text{zero}} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} + \left[{}^J\omega'_{K,\hat{\omega}'_L}\right] \begin{Bmatrix} \hat{\omega}'_{L1} \\ \hat{\omega}'_{L2} \\ \hat{\omega}'_{L3} \end{Bmatrix}$$

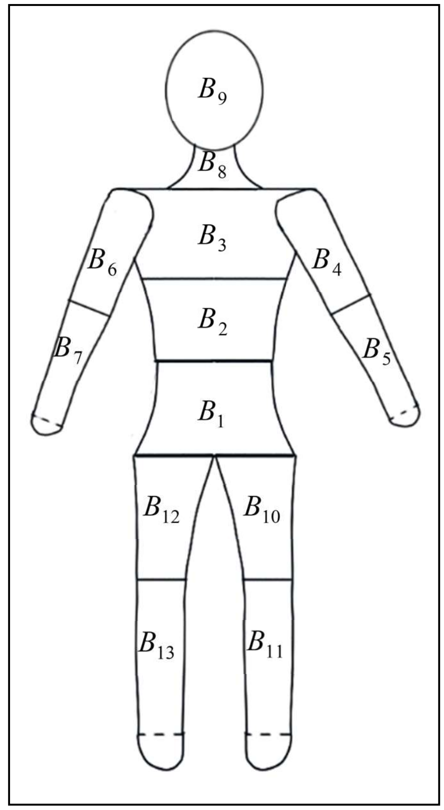
d) $$\left\{\omega'_K\right\} = \left[{}^JR_K\right]\left\{\omega'_J\right\} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\hat{\omega}'_K\right\} \triangleq \left[{}^R\omega'_{K,\omega'_J}\right]\left\{\omega'_J\right\} + \left[{}^R\omega'_{K,\hat{\omega}'_K}\right]\left\{\hat{\omega}'_K\right\} + \underbrace{\left[{}^R\omega'_{K,\hat{\omega}'_L}\right]}_{\text{zero}}\left\{\hat{\omega}'_L\right\}$$

$$\left\{\omega'_L\right\} = \left[{}^KR_L\right]\left[{}^JR_K\right]\left\{\omega'_J\right\} + \left[{}^KR_L\right]\left\{\hat{\omega}'_K\right\} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\hat{\omega}'_L\right\}$$
$$\triangleq \left[{}^R\omega'_{K,\omega'_J}\right]\left\{\omega'_J\right\} + \left[{}^R\omega'_{K,\hat{\omega}'_K}\right]\left\{\hat{\omega}'_K\right\} + \left[{}^R\omega'_{K,\hat{\omega}'_L}\right]\left\{\hat{\omega}'_L\right\}$$

2.9 The figure shows a thirteen-body model of the human body numbered using the numbering scheme presented in Unit 1. Body 1 is the lower torso, and it is the system reference body. The rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower-numbered body array for the system is as follows.

$$\mathcal{L}(1,\ldots,13)=(0,1,2,3,4,3,6,3,8,1,10,1,12)$$

The orientation of body 1 is defined relative to the fixed frame $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$, and the orientations of all the other bodies are defined relative to their adjacent, lower-numbered bodies. Using ***base frame components*** of the ***relative angular velocities*** of the bodies as generalized speeds, complete the following. The 3×1 vectors $\{\omega_K\}$ $(K=1,\ldots,13)$ contain the ***fixed frame components*** of the angular velocities of the bodies. The 3×1 vectors $\{\hat{\omega}_K\}$ $(K=1,\ldots,13)$ are of the ***base frame components*** of the angular velocities of the bodies ***relative*** to their ***base frames*** (lower-numbered bodies).



a) Define the ***fixed frame components*** of the ***angular velocities*** for all bodies in the system.
b) Combine the angular velocity components into a single 39×1 system matrix $\{\omega\}_{39\times1}$.
c) Define the ***fixed frame components*** of the ***partial angular velocities*** for all the bodies in the system.
d) Define a 3×39 partial angular velocity matrix for each body in the system.
e) Write the ***fixed frame components*** of the ***angular velocity*** of each body in terms of the ***system angular velocity matrix*** defined in part (b) and the ***partial angular velocity matrices*** defined in part (d).

Answers:

a)
$\{\omega_1\}=\{\hat{\omega}_1\}=\begin{bmatrix}\hat{\omega}_{11} & \hat{\omega}_{12} & \hat{\omega}_{13}\end{bmatrix}^T$ $\quad\{\omega_2\}=\{\omega_1\}+[R_1]^T\{\hat{\omega}_2\}$ $\quad\{\omega_3\}=\{\omega_2\}+[R_2]^T\{\hat{\omega}_3\}$

$\{\omega_4\}=\{\omega_3\}+[R_3]^T\{\hat{\omega}_4\}$ $\quad\{\omega_5\}=\{\omega_4\}+[R_4]^T\{\hat{\omega}_5\}$ $\quad\{\omega_6\}=\{\omega_3\}+[R_3]^T\{\hat{\omega}_6\}$

$\{\omega_7\}=\{\omega_6\}+[R_6]^T\{\hat{\omega}_7\}$ $\quad\{\omega_8\}=\{\omega_3\}+[R_3]^T\{\hat{\omega}_8\}$ $\quad\{\omega_9\}=\{\omega_8\}+[R_8]^T\{\hat{\omega}_9\}$

$\{\omega_{10}\}=\{\omega_1\}+[R_1]^T\{\hat{\omega}_{10}\}$ $\quad\{\omega_{11}\}=\{\omega_{10}\}+[R_{10}]^T\{\hat{\omega}_{11}\}$ $\quad\{\omega_{12}\}=\{\omega_1\}+[R_1]^T\{\hat{\omega}_{12}\}$

$\{\omega_{13}\}=\{\omega_{12}\}+[R_{12}]^T\{\hat{\omega}_{13}\}$

b) $\{\omega\}_{39\times1}=\begin{bmatrix}(\hat{\omega}_1)_1 & (\hat{\omega}_1)_2 & (\hat{\omega}_1)_3 & (\hat{\omega}_2)_1 & (\hat{\omega}_2)_2 & (\hat{\omega}_2)_3 & \cdots & (\hat{\omega}_{13})_1 & (\hat{\omega}_{13})_2 & (\hat{\omega}_{13})_3\end{bmatrix}^T$

c) Body 1: $\left[{}^R\omega_{1,\hat{\omega}_K}\right]=[0]_{3\times3}\quad(K\neq1)$ $\qquad\left[{}^R\omega_{1,\hat{\omega}_1}\right]=[I]_{3\times3}$

Body 2: $\left[{}^R\omega_{2,\hat{\omega}_K}\right]=\left[{}^R\omega_{1,\hat{\omega}_K}\right]\quad(K\neq2)$ $\qquad\left[{}^R\omega_{2,\hat{\omega}_2}\right]=[R_1]^T_{3\times3}$

Body 3: $\left[{}^{R}\omega_{3,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{2,\hat{\omega}_K}\right] \quad (K \neq 3)$ $\qquad \left[{}^{R}\omega_{3,\hat{\omega}_3}\right] = \left[R_2\right]^T_{3\times 3}$

Body 4: $\left[{}^{R}\omega_{4,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{3,\hat{\omega}_K}\right] \quad (K \neq 4)$ $\qquad \left[{}^{R}\omega_{4,\hat{\omega}_4}\right] = \left[R_3\right]^T_{3\times 3}$

Body 5: $\left[{}^{R}\omega_{5,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{4,\hat{\omega}_K}\right] \quad (K \neq 5)$ $\qquad \left[{}^{R}\omega_{5,\hat{\omega}_5}\right] = \left[R_4\right]^T_{3\times 3}$

Body 6: $\left[{}^{R}\omega_{6,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{3,\hat{\omega}_K}\right] \quad (K \neq 6)$ $\qquad \left[{}^{R}\omega_{6,\hat{\omega}_6}\right] = \left[R_3\right]^T_{3\times 3}$

Body 7: $\left[{}^{R}\omega_{7,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{6,\hat{\omega}_K}\right] \quad (K \neq 7)$ $\qquad \left[{}^{R}\omega_{7,\hat{\omega}_7}\right] = \left[R_6\right]^T_{3\times 3}$

Body 8: $\left[{}^{R}\omega_{8,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{3,\hat{\omega}_K}\right] \quad (K \neq 8)$ $\qquad \left[{}^{R}\omega_{8,\hat{\omega}_8}\right] = \left[R_3\right]^T_{3\times 3}$

Body 9: $\left[{}^{R}\omega_{9,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{8,\hat{\omega}_K}\right] \quad (K \neq 9)$ $\qquad \left[{}^{R}\omega_{9,\hat{\omega}_9}\right] = \left[R_8\right]^T_{3\times 3}$

Body 10: $\left[{}^{R}\omega_{10,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{1,\hat{\omega}_K}\right] \quad (K \neq 10)$ $\qquad \left[{}^{R}\omega_{10,\hat{\omega}_{10}}\right] = \left[R_1\right]^T_{3\times 3}$

Body 11: $\left[{}^{R}\omega_{11,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{10,\hat{\omega}_K}\right] \quad (K \neq 11)$ $\qquad \left[{}^{R}\omega_{11,\hat{\omega}_{11}}\right] = \left[R_{10}\right]^T_{3\times 3}$

Body 12: $\left[{}^{R}\omega_{12,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{1,\hat{\omega}_K}\right] \quad (K \neq 12)$ $\qquad \left[{}^{R}\omega_{12,\hat{\omega}_{12}}\right] = \left[R_1\right]^T_{3\times 3}$

Body 13: $\left[{}^{R}\omega_{13,\hat{\omega}_K}\right] = \left[{}^{R}\omega_{12,\hat{\omega}_K}\right] \quad (K \neq 13)$ $\qquad \left[{}^{R}\omega_{13,\hat{\omega}_{13}}\right] = \left[R_{12}\right]^T_{3\times 3}$

d) $\left[{}^{R}\omega_{K,\omega}\right]_{3\times 39} \quad (K = 1,\ldots,13)$

$\searrow$

$$
\begin{array}{l}
K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \\ K=9 \rightarrow \\ K=10 \rightarrow \\ K=11 \rightarrow \\ K=12 \rightarrow \\ K=13 \rightarrow
\end{array}
\begin{bmatrix}
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [R_3]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [R_3]^T & [R_4]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [R_3]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [R_3]^T & [R_6]^T & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [R_3]^T & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [R_3]^T & [R_8]^T & [0] & [0] & [0] & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [0] & [0] & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [R_{10}]^T & [0] & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [R_{12}]^T
\end{bmatrix}
$$

As noted in Example 3, the coordinate transformation matrices are constructed using the relative transformation matrices.

e) $\left\{\omega_K\right\}_{3\times1} = \left[{}^R\omega_{K,\omega}\right]_{3\times39}\left\{\omega\right\}_{39\times1}$ $\quad (K=1,\ldots,13)$

2.10 Consider again the thirteen-body model of the human body of Exercise 2.9. Using ***body frame components*** of the ***relative angular velocities*** of the bodies as generalized speeds, complete the following. The 3×1 vectors $\left\{\omega'_K\right\}$ $(K=1,\ldots,13)$ contain the ***body frame components*** of the angular velocities of the bodies. The 3×1 vectors $\left\{\hat{\omega}'_K\right\}$ $(K=1,\ldots,13)$ contain the ***body frame components*** of the angular velocities of the bodies ***relative*** to their ***base frames*** (lower-numbered bodies).

a) Define the ***body frame components*** of the ***angular velocities*** for all bodies in the system.

b) Combine the angular velocity components into a single 39×1 system matrix $\left\{\omega'\right\}_{39\times1}$.

c) Define the ***body frame components*** of the ***partial angular velocities*** for all the bodies in the system.

d) Define a 3×39 partial angular velocity matrix for each body in the system.

e) Write the ***body frame components*** of the ***angular velocity*** of each body in terms of the ***system angular velocity matrix*** defined in part (b) and the ***partial angular velocity matrices*** defined in part (d).

Answers:

a) $\left\{\omega'_1\right\} = \left\{\hat{\omega}'_1\right\} = \left[\hat{\omega}'_{11} \quad \hat{\omega}'_{12} \quad \hat{\omega}'_{13}\right]^T$ $\quad \left\{\omega'_2\right\} = \left[{}^1R_2\right]\left\{\omega'_1\right\} + \left\{\hat{\omega}'_2\right\}$ $\quad \left\{\omega'_3\right\} = \left[{}^2R_3\right]\left\{\omega'_2\right\} + \left\{\hat{\omega}'_3\right\}$

$\left\{\omega'_4\right\} = \left[{}^3R_4\right]\left\{\omega'_3\right\} + \left\{\hat{\omega}'_4\right\}$ $\quad \left\{\omega'_5\right\} = \left[{}^4R_5\right]\left\{\omega'_4\right\} + \left\{\hat{\omega}'_5\right\}$ $\quad \left\{\omega'_6\right\} = \left[{}^3R_6\right]\left\{\omega'_3\right\} + \left\{\hat{\omega}'_6\right\}$

$\left\{\omega'_7\right\} = \left[{}^6R_7\right]\left\{\omega'_6\right\} + \left\{\hat{\omega}'_7\right\}$ $\quad \left\{\omega'_8\right\} = \left[{}^3R_8\right]\left\{\omega'_3\right\} + \left\{\hat{\omega}'_8\right\}$ $\quad \left\{\omega'_9\right\} = \left[{}^8R_9\right]\left\{\omega'_8\right\} + \left\{\hat{\omega}'_9\right\}$

$\left\{\omega'_{10}\right\} = \left[{}^1R_{10}\right]\left\{\omega'_1\right\} + \left\{\hat{\omega}'_{10}\right\}$ $\quad \left\{\omega'_{11}\right\} = \left[{}^{10}R_{11}\right]\left\{\omega'_{10}\right\} + \left\{\hat{\omega}'_{11}\right\}$ $\quad \left\{\omega'_{12}\right\} = \left[{}^1R_{12}\right]\left\{\omega'_1\right\} + \left\{\hat{\omega}'_{12}\right\}$

$\left\{\omega'_{13}\right\} = \left[{}^{12}R_{13}\right]\left\{\omega'_{12}\right\} + \left\{\hat{\omega}'_{13}\right\}$

b) $\left\{\omega'\right\}_{39\times1} = \left[\left(\hat{\omega}'_1\right)_1 \quad \left(\hat{\omega}'_1\right)_2 \quad \left(\hat{\omega}'_1\right)_3 \quad \left(\hat{\omega}'_2\right)_1 \quad \left(\hat{\omega}'_2\right)_2 \quad \left(\hat{\omega}'_2\right)_3 \quad \cdots \quad \left(\hat{\omega}'_{13}\right)_1 \quad \left(\hat{\omega}'_{13}\right)_2 \quad \left(\hat{\omega}'_{13}\right)_3\right]^T$

c) In the results given below, $\left[I\right]_{3\times3}$ is the 3×3 identity matrix, and $\left[0\right]_{3\times3}$ is the 3×3 zero matrix.

Body 1: $\left[{}^R\omega'_{1,\hat{\omega}'_K}\right] = \left[0\right]_{3\times3}$ $\quad (K\neq1)$ $\qquad \left[{}^R\omega'_{1,\hat{\omega}'_1}\right] = \left[I\right]_{3\times3}$

Body 2: $\left[{}^R\omega'_{2,\hat{\omega}'_K}\right] = \left[{}^1R_2\right]\left[{}^R\omega'_{1,\hat{\omega}'_K}\right]$ $\quad (K\neq2)$ $\qquad \left[{}^R\omega'_{2,\hat{\omega}'_2}\right] = \left[I\right]_{3\times3}$

Body 3: $\left[{}^R\omega'_{3,\hat{\omega}'_K}\right] = \left[{}^2R_3\right]\left[{}^R\omega'_{2,\hat{\omega}'_K}\right]$ $\quad (K\neq3)$ $\qquad \left[{}^R\omega'_{3,\hat{\omega}'_3}\right] = \left[I\right]_{3\times3}$

Body 4: $\left[{}^R\omega'_{4,\hat{\omega}'_K}\right] = \left[{}^3R_4\right]\left[{}^R\omega'_{3,\hat{\omega}'_K}\right]$ $\quad (K\neq4)$ $\qquad \left[{}^R\omega'_{4,\hat{\omega}'_4}\right] = \left[I\right]_{3\times3}$

Body 5: $\left[{}^R\omega'_{5,\hat{\omega}'_K}\right] = \left[{}^4R_5\right]\left[{}^R\omega'_{4,\hat{\omega}'_K}\right]$ $\quad (K\neq5)$ $\qquad \left[{}^R\omega'_{5,\hat{\omega}'_5}\right] = \left[I\right]_{3\times3}$

Body 6: $\left[{}^R\omega'_{6,\hat{\omega}'_K}\right] = \left[{}^3R_6\right]\left[{}^R\omega'_{3,\hat{\omega}'_K}\right]$ $\quad (K\neq6)$ $\qquad \left[{}^R\omega'_{6,\hat{\omega}'_6}\right] = \left[I\right]_{3\times3}$

Body 7: $\left[{}^R\omega'_{7,\hat{\omega}'_K}\right] = \left[{}^6R_7\right]\left[{}^R\omega'_{6,\hat{\omega}'_K}\right]$ $\quad (K\neq7)$ $\qquad \left[{}^R\omega'_{7,\hat{\omega}'_7}\right] = \left[I\right]_{3\times3}$

Body 8: $\boxed{\left[{}^{R}\omega'_{8,\hat{\omega}'_K}\right]=\left[{}^{3}R_{8}\right]\left[{}^{R}\omega'_{3,\hat{\omega}'_K}\right]\quad (K\neq 8)}$ $\qquad$ $\boxed{\left[{}^{R}\omega'_{8,\hat{\omega}'_8}\right]=\left[I\right]_{3\times 3}}$

Body 9: $\boxed{\left[{}^{R}\omega'_{9,\hat{\omega}'_K}\right]=\left[{}^{8}R_{9}\right]\left[{}^{R}\omega'_{8,\hat{\omega}'_K}\right]\quad (K\neq 9)}$ $\qquad$ $\boxed{\left[{}^{R}\omega'_{9,\hat{\omega}'_9}\right]=\left[I\right]_{3\times 3}}$

Body 10: $\boxed{\left[{}^{R}\omega'_{10,\hat{\omega}'_K}\right]=\left[{}^{1}R_{10}\right]\left[{}^{R}\omega'_{1,\hat{\omega}'_K}\right]\quad (K\neq 10)}$ $\qquad$ $\boxed{\left[{}^{R}\omega'_{10,\hat{\omega}'_{10}}\right]=\left[I\right]_{3\times 3}}$

Body 11: $\boxed{\left[{}^{R}\omega'_{11,\hat{\omega}'_K}\right]=\left[{}^{10}R_{11}\right]\left[{}^{R}\omega'_{10,\hat{\omega}'_K}\right]\quad (K\neq 11)}$ $\qquad$ $\boxed{\left[{}^{R}\omega'_{11,\hat{\omega}'_{11}}\right]=\left[I\right]_{3\times 3}}$

Body 12: $\boxed{\left[{}^{R}\omega'_{12,\hat{\omega}'_K}\right]=\left[{}^{1}R_{12}\right]\left[{}^{R}\omega'_{1,\hat{\omega}'_K}\right]\quad (K\neq 12)}$ $\qquad$ $\boxed{\left[{}^{R}\omega'_{12,\hat{\omega}'_{12}}\right]=\left[I\right]_{3\times 3}}$

Body 13: $\boxed{\left[{}^{R}\omega'_{13,\hat{\omega}'_K}\right]=\left[{}^{12}R_{13}\right]\left[{}^{R}\omega'_{12,\hat{\omega}'_K}\right]\quad (K\neq 13)}$ $\qquad$ $\boxed{\left[{}^{R}\omega'_{13,\hat{\omega}'_{13}}\right]=\left[I\right]_{3\times 3}}$

d) $\left[{}^{R}\omega'_{K,\omega'}\right]_{3\times 39}\quad (K=1,\ldots,13)$

↘

$$
\begin{array}{ll}
K=1 \rightarrow \\
K=2 \rightarrow \\
K=3 \rightarrow \\
K=4 \rightarrow \\
K=5 \rightarrow \\
K=6 \rightarrow \\
K=7 \rightarrow \\
K=8 \rightarrow \\
K=9 \rightarrow \\
K=10 \rightarrow \\
K=11 \rightarrow \\
K=12 \rightarrow \\
K=13 \rightarrow
\end{array}
\begin{bmatrix}
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{2}] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{3}] & [{}^{2}R_{3}] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{4}] & [{}^{2}R_{4}] & [{}^{3}R_{4}] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{5}] & [{}^{2}R_{5}] & [{}^{3}R_{5}] & [{}^{4}R_{5}] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{6}] & [{}^{2}R_{6}] & [{}^{3}R_{6}] & [0] & [0] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{7}] & [{}^{2}R_{7}] & [{}^{3}R_{7}] & [0] & [0] & [{}^{6}R_{7}] & [I] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{8}] & [{}^{2}R_{8}] & [{}^{3}R_{8}] & [0] & [0] & [0] & [0] & [I] & [0] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{9}] & [{}^{2}R_{9}] & [{}^{3}R_{9}] & [0] & [0] & [0] & [0] & [{}^{8}R_{9}] & [I] & [0] & [0] & [0] & [0] \\
[{}^{1}R_{10}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [I] & [0] & [0] & [0] \\
[{}^{1}R_{11}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [{}^{10}R_{11}] & [I] & [0] & [0] \\
[{}^{1}R_{12}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [I] & [0] \\
[{}^{1}R_{13}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [{}^{12}R_{13}] & [I]
\end{bmatrix}
$$

As noted in Example 4, the transformation matrices are constructed from the individual relative transformation matrices.

e) $\boxed{\left\{\omega'_K\right\}_{3\times 1}=\left[{}^{R}\omega'_{K,\omega'}\right]_{3\times 39}\left\{\omega'\right\}_{39\times 1}}\quad (K=1,\ldots,13)$

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