Elementary Dynamics Example #3: (Rectilinear Motion)

Given: $a(v) = g - c\,v \quad (a(v) \geq 0)$ … the ***acceleration*** of an object falling in a fluid.

Initial condition: $v(0) = 0$

Find: $v(t)$ … the ***velocity*** of the particle as a ***function of time***

Solution:

$$a(v) = \frac{dv}{dt} = g - c\,v \quad \Rightarrow \quad \frac{dv}{g - c\,v} = dt \quad \Rightarrow \quad \tfrac{-1}{c}\int \frac{-c\,dv}{g - c\,v} = \int dt$$ … using indefinite integrals

$$\boxed{\tfrac{-1}{c}\ln(g - c\,v) = t + D}$$

Aside: $\displaystyle\int \frac{f'(x)\,dx}{f(x)} = \ln\big(f(x)\big)$

Applying the ***initial condition***, $v(0) = 0$, gives $\boxed{D = \tfrac{-1}{c}\ln(g)}$

Substituting the value of the constant D into the first boxed equation and ***simplifying***:

$$\tfrac{-1}{c}\ln(g - c\,v) = t + \tfrac{-1}{c}\ln(g) \quad \Rightarrow \quad \ln(g - c\,v) = \ln(g) - c\,t \quad \Rightarrow \quad g - c\,v = \exp\big(\ln(g) - c\,t\big)$$

$$g - c\,v = \exp\big(\ln(g)\big)\cdot\exp\big(-c\,t\big) = g\,e^{-ct} \quad \Rightarrow \quad \boxed{v(t) = \tfrac{g}{c}\big(1 - e^{-ct}\big)}$$

The function $v(t)$ starts at ***zero*** and increases exponentially to the ***final value*** of $\frac{g}{c}$. The ***larger*** the value of the coefficient c, the ***faster*** the function increases toward $\frac{g}{c}$. The plot below shows a plot a $v(t)$ for $c = g = 1$.