Introductory Control Systems
Mathematical Models of Physical Systems

The responses of some physical systems are ***relatively simple***, and the systems needed to control them can be designed using ***trial-and-error***. For more ***complex system responses***, however, a more sophisticated design approach is necessary. This approach is based on developing ***mathematical*** (numerical) ***models*** that describe how the process and its associated components respond. These ***quantitative mathematical models*** can be based on experimental observations or known physical principles.

A mathematical model can consist of ***differential*** and/or ***algebraic equations***. The solution of these equations ***describes*** the ***dynamics*** of the system, that is, how the system responds to its expected input. A system can consist of a ***single component*** or of ***many different types of components*** – mechanical, electrical, hydraulic, thermal, etc.

A mathematical model can be ***linear*** or ***nonlinear*** depending on the system and the range of operation being modeled. If a system is nonlinear, it may be possible to ***linearize*** the model before applying linear analysis to the system. The extent to which this approach is applicable depends on the ***strength*** and ***type*** of ***nonlinearities***.

Mathematical models can be developed using ***physical principles***. Using this approach, the analyst writes the ***differential*** and/or ***algebraic equations*** that are thought to describe the system dynamics. Laplace transforms can then used to convert the ***differential equations*** into ***transfer functions***. This approach is ***limited*** by the ***analyst's ability*** to: 1) ***describe*** the ***physics*** of the system (especially for complex systems), and 2) ***estimate*** the important ***parameters***. As an example of mathematical modeling, the spring-mass-damper system of Fig. 1 is analyzed below.

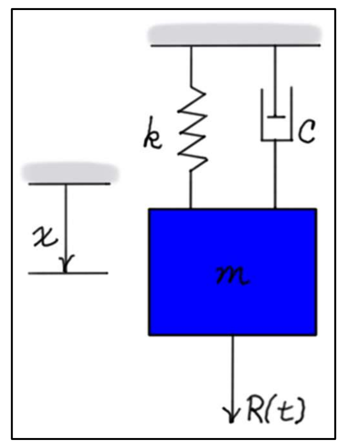
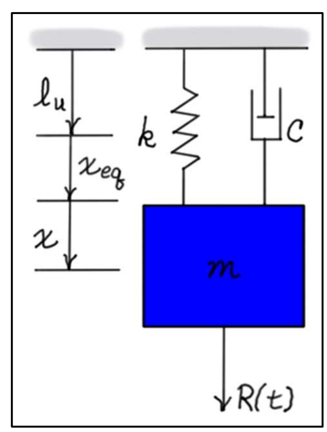


Fig. 1. Spring-Mass-Damper System

Example: Spring-Mass-Damper System

<u>Nomenclature</u>



m : mass of the block

k : spring stiffness

c : coefficient of the damper

$R(t)$: external force (input)

ℓ_u : unstretched length of spring

x_{eq} : equilibrium position of mass (hanging position)

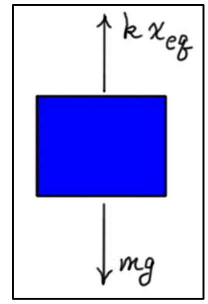
x : position of mass ***relative to*** the equilibrium position (output)

$\dot{x}$: velocity of mass

$\ddot{x}$: acceleration of mass

As indicated in the adjacent diagram, the position of the mass relative to the fixed upper support is given by the sum of three quantities – 1) ℓ_u the unstretched length of the spring, 2) x_{eq} the static position of the mass, and 3) x the position of the mass relative to the static equilibrium position. Measuring a system's response away from an equilibrium position is very common in control system analysis. These positions are often called nominal positions (conditions) or set points.

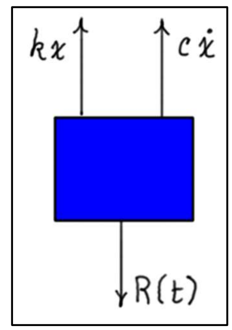
Static Equilibrium Position



Applying the equations of ***static equilibrium*** to the adjacent free body diagram, the ***equilibrium position*** of the system under its own weight can be written as follows.

$$\boxed{x_{eq} = mg / k} \qquad (1)$$

Equation of Motion about the Equilibrium Position



Applying Newton's second law to the adjacent free body diagram, the ***differential equation of motion*** can be written as follows.

$$\boxed{m\,\ddot{x} \;+\; c\,\dot{x} \;+\; k\,x = R(t)} \qquad (2)$$

Here, the variable x is measured from the ***equilibrium position***. Note that ***static forces*** are ***not present*** in this equation.

The solution of this equation describes the ***forced response*** of the system. The ***free response*** of the system is described by solving the equation with $R(t) \equiv 0$. In both cases, the initial conditions $x(0)$ and $\dot{x}(0)$ must be specified to find a unique solution.

System Parameters

The system mass m, spring stiffness k, and damping coefficient c are the ***system's parameters***. The first two are usually easier to measure (or estimate) than the third. For the model to be useful, reasonable estimates of these parameters are necessary.

System Characteristic Equation and Response Type

The ***differential equation*** of ***motion*** represents a ***mathematical model*** of the system. It has ***three types*** of ***solutions*** depending on the values of the parameters m, c, and k. The type (or ***character***) of solution is determined by the roots of the ***characteristic equation*** of the system. For the spring-mass-damper system, it can be shown that the characteristic equation can be written as follows.

$$\boxed{s^2 + (c/m)s + (k/m) = 0} \qquad \text{or} \qquad \boxed{s^2 + (2\zeta\omega_n)s + \omega_n^2 = 0}$$

Here,

$\boxed{\omega_n = \sqrt{\dfrac{k}{m}}}$ is the ***natural frequency*** of the system

$$\boxed{\zeta = \frac{c}{2\sqrt{mk}}}$$ is the ***damping ratio***

In general, the ***roots*** of the ***characteristic equation*** can be written in the form

$$\boxed{s_{1,2} = -\zeta\omega_n \ \pm \ \omega_n\sqrt{\zeta^2 - 1}}$$

These roots may be ***real*** or ***complex*** depending on the value of ζ . The following table shows the ***three types*** of possible ***motion***.

Case	Type of Roots	Type of Motion	Form of Solution
$\zeta < 1$	Complex conjugates	Under-damped	$x(t) = A\,e^{-\zeta\,\omega_n\,t}\left(\cos\left(\omega_{\mathrm{d}}\,t + \varphi\right)\right)$
$\zeta > 1$	Real, unequal	Over-damped	$x(t) = A\,e^{s_1\,t} + B\,e^{s_2\,t}$
$\zeta = 1$	Real, equal	Critically damped	$x(t) = A\,e^{-\zeta\,\omega_n\,t} + B\,t\,e^{-\zeta\,\omega_n\,t}$