Introductory Control Systems
Linearization of System Models

The ***differential*** or ***algebraic*** equation that models the behavior of a system may be ***linear*** or ***nonlinear***. If it is nonlinear, it may be possible to ***linearize*** the equation about some ***nominal operating condition*** (or ***set point***). The focus here is on linearization of ***algebraic models only***.

Linearization of Functions of a Single Variable

If the output of a system (y) is related to the input of the system (x) with a nonlinear algebraic function $y = f(x)$, an approximate linear model of the system behavior can be developed using a Taylor series. First, identify a set point (nominal condition), and then expand $f(x)$ in a ***Taylor series*** around that point. Referring to the set point as an ***equilibrium position*** x_{eq}, write

$$f\left(x_{eq} + \Delta x\right) = f\left(x_{eq}\right) + \Delta x \left[\frac{df}{dx}\right]_{x=x_{eq}} + \frac{(\Delta x)^2}{2}\left[\frac{d^2 f}{dx^2}\right]_{x=x_{eq}} + \cdots$$

$$\approx f\left(x_{eq}\right) + \Delta x \left[\frac{df}{dx}\right]_{x=x_{eq}}$$

Here Δx represents an ***excursion*** or ***deviation*** from the equilibrium position (set point). If the ***excursions*** are ***small***, then the approximation as stated in the second equation can be used. In this latter case, write

$$\boxed{\Delta f(x) = f\left(x_{eq} + \Delta x\right) - f\left(x_{eq}\right) = m\,\Delta x}$$

Here,

$$\boxed{m = \left.\frac{df}{dx}\right|_{x=x_{eq}}}$$

This is a ***linear*** relationship between ***changes*** in *f* and ***changes*** in *x*.

Note: It is up to the analyst to determine the ***range*** of Δx over which the linear model is ***acceptably accurate***. This range could be ***quite large*** for ***mildly nonlinear*** functions, but it could be ***quite small*** for ***strongly nonlinear*** functions.

Linearization of Functions of Many Variables

Given a nonlinear function $y = f(x_1, x_2, \ldots, x_n) = f\left(\underset{\sim}{x}\right)$, expand $f(\underset{\sim}{x})$ in a ***Taylor series*** around the ***equilibrium position***, say $\underset{\sim}{x}_{eq} = \left((x_1)_{eq}, (x_2)_{eq}, \ldots, (x_n)_{eq}\right)$

$$f\left(\underset{\sim}{x}_{eq} + \Delta \underset{\sim}{x}\right) = f\left(\underset{\sim}{x}_{eq}\right) + \sum_{i=1}^{n} \Delta x_i \left[\frac{\partial f}{\partial x_i}\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}} + \cdots$$

$$\approx f\left(\underset{\sim}{x}_{eq}\right) + \sum_{i=1}^{n} \Delta x_i \left[\frac{\partial f}{\partial x_i}\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}}$$

Here the vector $\Delta \underset{\sim}{x} = \left(\Delta x_1, \Delta x_2, \ldots, \Delta x_n\right)$ represents an ***excursion*** from the ***equilibrium position***. As before, if the excursions are ***small***, then the approximation as stated in the second equation can be used. In this latter case, write

$$\boxed{\Delta f\left(\underset{\sim}{x}\right) = f\left(\underset{\sim}{x}_{eq} + \Delta \underset{\sim}{x}\right) - f\left(\underset{\sim}{x}_{eq}\right) = \sum_{i=1}^{n} m_i \, \Delta x_i}$$

Here,

$$\boxed{m_i = \left.\frac{\partial f}{\partial x_i}\right|_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}}}$$

This is a ***linear*** relationship between ***changes*** in *f* and ***changes*** in the elements of vector $\underset{\sim}{x}$. As before, it is up to the analyst to determine acceptable ranges of the $\Delta x_i \; (i = 1, \ldots n)$.