Elementary Dynamics
Curvilinear Motion – Radial and Transverse Components

Radial and Transverse Components

Another way to describe the motion of P as it moves along a curved path is to use **radial** and **transverse** components. Here, we define the unit vector $\underset{\sim}{e}_r$ to point radially outward from O to P. The **position vector** of P can then be written as

$$\boxed{\underset{\sim}{r} = r\,\underset{\sim}{e}_r}$$

To find an expression for the **velocity** of P, we differentiate (using the product rule)

$$\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\underset{\sim}{e}}_r = \dot{r}\underset{\sim}{e}_r + r\left(\dot{\theta}\underset{\sim}{k}\times\underset{\sim}{e}_r\right) = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta \quad \Rightarrow \quad \boxed{\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta}$$

To find an expression for the **acceleration** of P, differentiate again using the product rule.

$$\underset{\sim}{a} = \ddot{r}\,\underset{\sim}{e}_r + \dot{r}\,\dot{\underset{\sim}{e}}_r + \dot{r}\dot{\theta}\,\underset{\sim}{e}_\theta + r\ddot{\theta}\,\underset{\sim}{e}_\theta + r\dot{\theta}\,\dot{\underset{\sim}{e}}_\theta$$

Here,

$$\dot{\underset{\sim}{e}}_r = \dot{\theta}\underset{\sim}{k}\times\underset{\sim}{e}_r = \dot{\theta}\underset{\sim}{e}_\theta \quad \text{and} \quad \dot{\underset{\sim}{e}}_\theta = \dot{\theta}\underset{\sim}{k}\times\underset{\sim}{e}_\theta = -\dot{\theta}\underset{\sim}{e}_r$$

Using these two results in the expression for the acceleration and collecting terms gives

$$\boxed{\underset{\sim}{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underset{\sim}{e}_\theta}$$
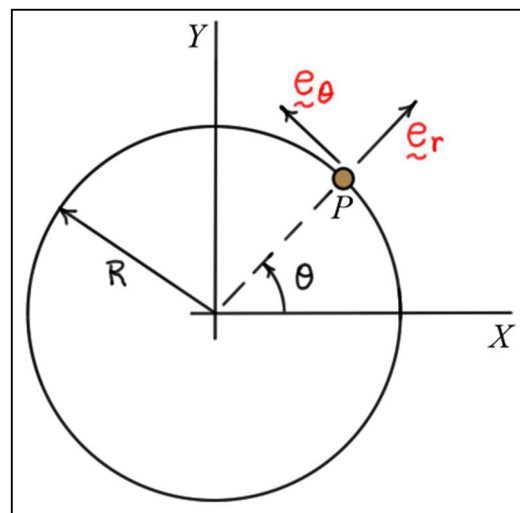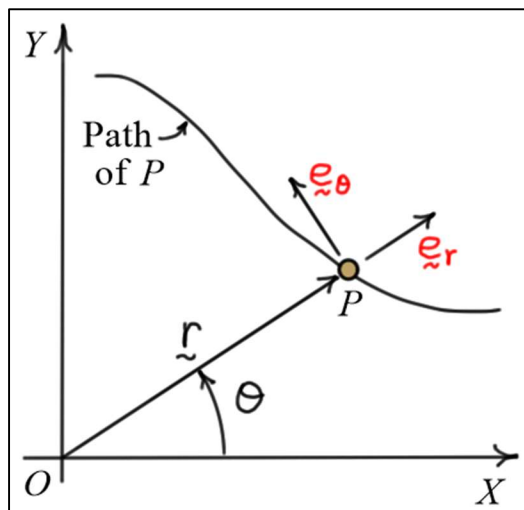
Special Case: Circular Motion

During circular motion the distance between O and P remains **constant** and the transverse unit vector $\underset{\sim}{e}_\theta$ is **tangent** to the path.

$$r = R = \text{constant} \quad \Rightarrow \dot{r} = \ddot{r} = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \underset{\sim}{v} = R\dot{\theta}\underset{\sim}{e}_\theta \\ \underset{\sim}{a} = \left(-R\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(R\ddot{\theta}\right)\underset{\sim}{e}_\theta \end{array}}$$

Cylindrical Components

Cylindrical components represent the extension of the concept of *radial* and *transverse* components to three dimensions. The projection of P onto the XY plane is tracked with radial and transverse directions in that plane. Motion of P perpendicular to the XY plane is tracked by the Cartesian (rectangular) coordinate z. In this case, write

$$\underset{\sim}{r} = r\underset{\sim}{e}_r + z\underset{\sim}{k}$$

$$\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta + \dot{z}\underset{\sim}{k}$$

$$\underset{\sim}{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underset{\sim}{e}_\theta + \ddot{z}\underset{\sim}{k}$$