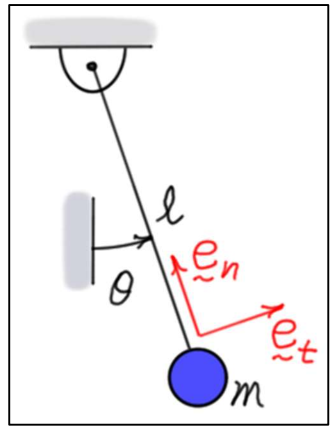
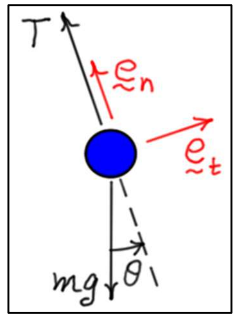
Introductory Control Systems
Equations of Motion of a Simple Pendulum and Accelerating Simple Pendulum

Simple Pendulum



- A ***simple pendulum*** with end mass m is shown in the figure to the right. The end mass is assumed to be connected to the support with a ***light***, ***slender***, ***rigid rod*** of length ℓ.
- The ***differential equation of motion*** of the pendulum can be found by summing forces in the tangential $(\underset{\sim}{e}_t)$ direction. Referring to the free body diagram, write



$$\boxed{\sum F_t = ma_t \quad \Rightarrow \quad -mg\sin(\theta) = m\ell\ddot{\theta}}$$

or

$$\boxed{\ddot{\theta} + \tfrac{g}{\ell}\sin(\theta) = 0} \qquad (1)$$

- The solution of this differential equation describes the movement of the pendulum for ***any initial conditions***.
- <u>Equilibrium Positions</u>: These are found by setting $\boxed{\ddot{\theta} = 0}$ in the differential equation of motion (1). Hence, the pendulum has ***two equilibrium positions***: $\boxed{\theta_{eq} = 0, \pi}$.
- <u>Linearized Equation of Motion</u>: $\boxed{\theta_{eq} = 0}$ To study ***small motions*** of the pendulum about the equilibrium position $\theta_{eq} = 0$, let $\boxed{\theta = \theta_{eq} + \Delta\theta}$ and linearize the equation of motion (1) about that position. This is done by linearizing the function $\boxed{f(\theta) = \sin(\theta)}$ about $\theta = 0$.

$$\boxed{\Delta f = \left(\frac{df}{d\theta}\right)_{\theta=0} \Delta\theta = 1 \cdot \Delta\theta = \Delta\theta}$$

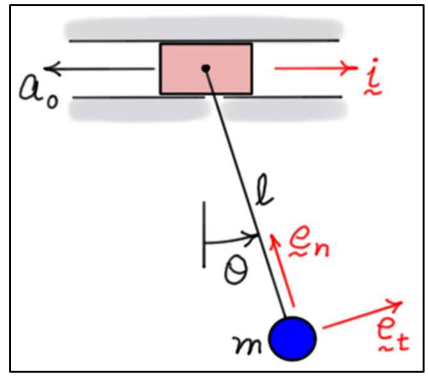
- Hence, the ***approximate linear equation of motion*** for small motions about $\theta = 0$ is

$$\boxed{\Delta\ddot{\theta} + \left(\tfrac{g}{\ell}\right)\Delta\theta = 0}$$

- The solution of this differential equation describes ***small movements*** of the pendulum about $\theta = 0$.

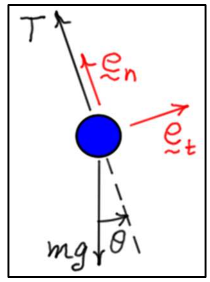
__Accelerating Simple Pendulum__

- An ***accelerating simple pendulum*** with end mass m is shown in the figure. The end mass is assumed to be connected to the accelerating base with a ***light***, ***slender***, ***rigid rod*** of length ℓ.



- The base ***accelerates*** at a ***constant*** rate a_0 to the ***left***. The acceleration of mass m may be found using the ***relative acceleration*** equation as follows:

$$\underset{\sim}{a}_m = \underset{\sim}{a}_A + \underset{\sim}{a}_{m/A} = -a_0\,\underset{\sim}{i} + \left(\ell\ddot{\theta}\underset{\sim}{e}_t + \ell\dot{\theta}^2\underset{\sim}{e}_n\right)$$
$$= \left(\ell\ddot{\theta} - a_0\cos(\theta)\right)\underset{\sim}{e}_t + \left(\ell\dot{\theta}^2 + a_0\sin(\theta)\right)\underset{\sim}{e}_n$$



- The ***differential equation of motion*** of the pendulum may be found by summing forces in the ***tangential*** ($\underset{\sim}{e}_t$) direction. Referring to the ***free body diagram***, write

$$\boxed{\sum F_t = ma_t \;\Rightarrow\; -mg\sin(\theta) = m\left(\ell\ddot{\theta} - a_0\cos(\theta)\right)} \Rightarrow \boxed{\ddot{\theta} + \left(\tfrac{g}{\ell}\sin(\theta) - \tfrac{a_0}{\ell}\cos(\theta)\right) = 0}$$

- The solution of this differential equation describes the movement of the pendulum for ***any initial conditions***.

- Equilibrium Positions: As before, these positions are found by setting $\boxed{\ddot{\theta} = 0}$. Hence, the accelerating pendulum has two equilibrium positions: $\boxed{\theta_{eq} = \tan^{-1}\left(\frac{a_0}{g}\right)}$. For example, if $a_0 = 0.5g$, then $\boxed{\theta_{eq} = \tan^{-1}(0.5) = 26.57^\circ, 206.57^\circ}$.

- Linearized Equation of Motion: $\boxed{a_0 = 0.5g}$ and $\boxed{\theta_{eq} = 26.57^\circ}$ To study ***small motion*** of the pendulum about $\theta_{eq} = 26.57^\circ$, let $\boxed{\theta = \theta_{eq} + \Delta\theta}$ and linearize the equation of motion about that position. This is done by linearizing the function $f(\theta) = \frac{g}{\ell}\left(\sin(\theta) - \frac{1}{2}\cos(\theta)\right)$ about $\theta_{eq} = 26.57^\circ$.

$$\boxed{\Delta f(\theta) = \left(\left.\frac{df}{d\theta}\right|_{\theta=\theta_{eq}}\right)\Delta\theta = \left(\tfrac{g}{\ell}\left(\cos(\theta) + \tfrac{1}{2}\sin(\theta)\right)\right)_{\theta=\theta_{eq}}\Delta\theta = \left(1.118\tfrac{g}{\ell}\right)\Delta\theta}$$

- The ***approximate linear equation of motion*** for small motion about $\theta = 26.57^{\text{o}}$ is

$$\boxed{\Delta\ddot{\theta} + \left(1.118\tfrac{g}{\ell}\right)\Delta\theta = 0}$$

- The solution of this differential equation describes ***small movements*** of the pendulum about $\theta = 26.57^{\text{o}}$.
- Note the ***multiplier*** of $\frac{g}{\ell}$ is 1.0 for the non-accelerating pendulum and 1.118 for the accelerating pendulum. It can be shown that this means the ***natural frequency*** of the ***accelerating pendulum*** is ***slightly higher*** that the ***non-accelerating pendulum***.