**Intermediate Dynamics**
Kinematics of a Point Moving on a Rigid Body

The kinematic analysis is now extended to include systems where interconnected bodies can *rotate* and *translate* relative to each other. In these cases, there is a need to describe the kinematics of points that are moving on (relative to) a rotating body. To analyze this motion, consider the figure shown at the right. Here,
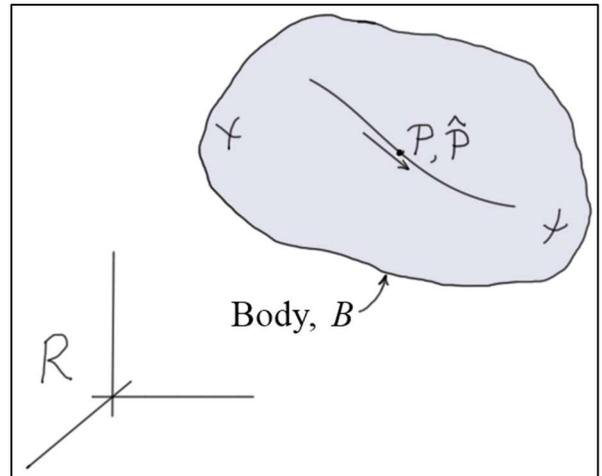
R : a fixed reference frame

B : a moving rigid body

P : a point *moving* on B

$\hat{P}$: a point *fixed* on B that *coincides* with P at this instant of time

The *velocity* and *acceleration* of P can be written as

$$\boxed{{}^{R}\underset{\sim}{v}_{P} = {}^{R}\underset{\sim}{v}_{\hat{P}} + {}^{B}\underset{\sim}{v}_{P}}$$

$$\boxed{{}^{R}\underset{\sim}{a}_{P} = {}^{R}\underset{\sim}{a}_{\hat{P}} + {}^{B}\underset{\sim}{a}_{P} + 2\left({}^{R}\underset{\sim}{\omega}_{B} \times {}^{B}\underset{\sim}{v}_{P}\right)}$$

Here, each of the terms are defined as follows.

$\quad {}^{B}\underset{\sim}{v}_{P}, {}^{B}\underset{\sim}{a}_{P}$: velocity and acceleration of P on B, assuming that B is fixed

$\quad {}^{R}\underset{\sim}{v}_{\hat{P}}, {}^{R}\underset{\sim}{a}_{\hat{P}}$: velocity and acceleration of $\hat{P}$ in R (recall that $\hat{P}$ is fixed on B)

$2\left({}^{R}\underset{\sim}{\omega}_{B} \times {}^{B}\underset{\sim}{v}_{P}\right)$: Coriolis acceleration of P

Note: The *velocity* and *acceleration* of $\hat{P}$ can be determined using the formulae for points fixed on rigid bodies. See notes on "Relative Kinematics of Two Points Fixed on a Rigid Body".

Derivation

The results shown above can easily be shown by using "the derivative rule". Consider the rigid body shown in the diagram at the right. Here,
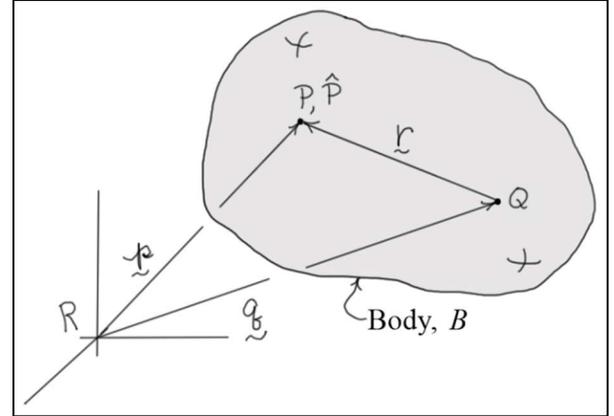

R: fixed reference frame
B: moving rigid body
P: point **moving** on B
$\hat{P}$: point **fixed** on B that **coincides** with P
Q: point **fixed** on B

The **velocity** of P can be found by **differentiating** its position vector as follows.

$$
\begin{aligned}
{}^{R}\underset{\sim}{v}_{P} &= \frac{{}^{R}d\underset{\sim}{p}}{dt} = \frac{{}^{R}d}{dt}\left(\underset{\sim}{q} + \underset{\sim}{r}\right) \\
&= \frac{{}^{R}d\underset{\sim}{q}}{dt} + \frac{{}^{R}d\underset{\sim}{r}}{dt} \\
&= {}^{R}\underset{\sim}{v}_{Q} + \frac{{}^{B}d\underset{\sim}{r}}{dt} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right) \\
&= {}^{R}\underset{\sim}{v}_{Q} + {}^{B}\underset{\sim}{v}_{P} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)
\end{aligned}
$$

Now, letting $\underset{\sim}{r} \to \underset{\sim}{0}$ (that is, letting Q be $\hat{P}$) the desired result is obtained. The **acceleration** of P can be found by **differentiating** the expression for its velocity.

$$
\begin{aligned}
{}^{R}\underset{\sim}{a}_{P} &= \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{Q} + {}^{B}\underset{\sim}{v}_{P} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)\right) \\
&= \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{Q}\right) + \left\{\frac{{}^{R}d}{dt}\left({}^{B}\underset{\sim}{v}_{P}\right)\right\} + \left\{\frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)\right\} \\
&= {}^{R}\underset{\sim}{a}_{Q} + \left\{\frac{{}^{B}d}{dt}\left({}^{B}\underset{\sim}{v}_{P}\right) + \left({}^{R}\underset{\sim}{\omega}_{B} \times {}^{B}\underset{\sim}{v}_{P}\right)\right\} + \left\{\left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left(\frac{{}^{B}d\underset{\sim}{r}}{dt} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)\right)\right\} \\
&= {}^{R}\underset{\sim}{a}_{Q} + \left\{{}^{B}\underset{\sim}{a}_{P} + \left({}^{R}\underset{\sim}{\omega}_{B} \times {}^{B}\underset{\sim}{v}_{P}\right)\right\} + \left\{\left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + \left({}^{R}\underset{\sim}{\omega}_{B} \times {}^{B}\underset{\sim}{v}_{P}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)\right\}
\end{aligned}
$$

Now, letting $\underset{\sim}{r} \to \underset{\sim}{0}$ (that is, letting Q be $\hat{P}$) the desired result is obtained.