Introductory Control Systems
Solutions to Linear, Ordinary Differential Equations Using Laplace Transforms

- ***Linear ordinary differential equations*** (ODE's) may be solved using ***Laplace transforms***. There are ***three*** steps in the solution process.
  1. The ***Laplace transform*** is used to ***convert*** the ***differential equation*** into an ***algebraic equation***.
  2. The algebraic equation is ***solved*** for the Laplace transform of the solution, and the concept of ***partial fractions*** is used to ***separate*** the solution into its elementary components.
  3. Finally, the ***inverse Laplace transform*** is taken of each of the elementary components to find the solution in the time domain.
- The ***definitions*** of the Laplace transform and the inverse Laplace transform are given below on page 2. Fortunately, however, ***tables*** of ***known transforms*** can be used to solve many differential equations.
- For most cases, the ***properties*** of the Laplace transform and ***tables*** of transforms of common functions are used to find solutions. A list of helpful properties of Laplace transforms is given below.

Properties of the Laplace Transform and Inverse Laplace Transforms

1. $\mathcal{L}(kf(t)) = k\mathcal{L}(f(t)) = kF(s)$ and $\mathcal{L}^{-1}(kF(s)) = kf(t)$
2. $\mathcal{L}(f_1(t) \pm f_2(t)) = F_1(s) \pm F_2(s)$ and $\mathcal{L}^{-1}(F_1(s) \pm F_2(s)) = f_1(t) \pm f_2(t)$
3. $\mathcal{L}\left(\dfrac{df}{dt}\right) = sF(s) - f(0)$ and $\mathcal{L}\left(\dfrac{d^2 f}{dt^2}\right) = s^2 F(s) - s\,f(0) - \dfrac{df}{dt}(0)$
4. $\mathcal{L}\left(\displaystyle\int_0^t f(t)dt\right) = \dfrac{F(s)}{s}$
5. Initial Value Theorem: $\lim\limits_{t\to 0}\left(f(t)\right) = \lim\limits_{s\to\infty}\left(s\,F(s)\right)$ (if the time limit exists)
6. Final Value Theorem: $\lim\limits_{t\to \infty}\left(f(t)\right) = \lim\limits_{s\to 0}\left(s\,F(s)\right)$ (if the time limit exists)

Note: Item 3 shows the formulae for the Laplace transforms of the first two derivatives of a function. There is a more general formula for the Laplace transform of the n^{th} derivative of a function that is used for differential equations of order three or higher.

Definitions of the Laplace Transform and Inverse Laplace Transform

The Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}\left(f(t)\right) = F(s) = \int_0^\infty f(t) e^{-st} dt$$

and the inverse Laplace transform is defined to be

$$\mathcal{L}^{-1}\left(F(s)\right) = f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

where s is a complex-valued variable in the Laplace domain.