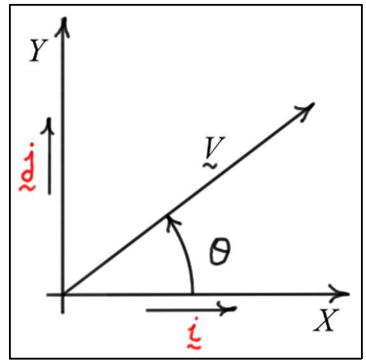
Elementary Engineering Mathematics

Application of Two-Dimensional Vectors – Statics, Mechanics of Materials, Dynamics

Scalars and Vectors

A ***scalar*** is a quantity represented by a ***positive*** or ***negative number***. It contains a magnitude (its absolute value) and a sign. They are sometimes called ***one-dimensional vectors***, because the sign refers to the direction along a single axis. Examples include length, area, volume, mass, pressure, and temperature.



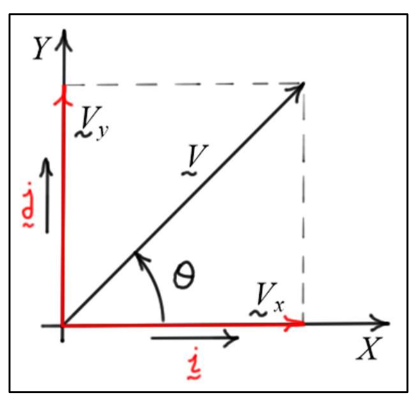
A two-dimensional (2D) vector is represented by both a ***magnitude*** and a ***direction*** related to ***two reference axes***. Usually, the reference axes (X and Y) are perpendicular to each other. In applications, vectors can be categorized as ***fixed*** or ***free***. A ***fixed*** vector is defined to be anchored at a specific point, whereas ***free*** vectors can be located anywhere without changing their meaning. Whether a vector is thought to be fixed or free depends on the quantity the vector represents.

For example, if vector $\underset{\sim}{V}$ in the diagram represents a ***force*** acting on some object, it is a ***fixed*** vector, because its ***point*** of ***application*** is important. In contrast, consider vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ shown in the diagram. These vectors have ***unit magnitude*** and point along the X and Y axes, respectively. They are called ***unit vectors*** and are used to define ***directions*** of interest. Since their point of origin is not important, they are ***free*** vectors. The ***mathematical representation*** of a vector does not indicate whether it is fixed or free, so we must be mindful of this as we use them.

Given: The magnitude $\left|\underset{\sim}{V}\right|$ and direction θ of vector $\underset{\sim}{V}$

Find: The X and Y ***components*** of $\underset{\sim}{V}$

Solution:



The diagram shows the X and Y ***components*** of $\underset{\sim}{V}$ labeled as $\underset{\sim}{V}_x$ and $\underset{\sim}{V}_y$. These components are also two-dimensional vectors, and their magnitudes are given by right-triangle trigonometry. Their directions are along $\underset{\sim}{i}$ and $\underset{\sim}{j}$ directions, respectively.

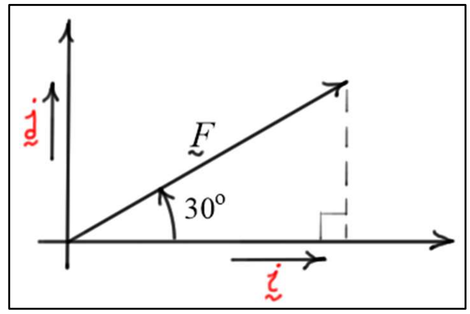
Vector $\underset{\sim}{V}$ is the sum of these two vector components.

$$\underset{\sim}{V}_x = |\underset{\sim}{V}|\cos(\theta)\,\underset{\sim}{i} \qquad \underset{\sim}{V}_y = |\underset{\sim}{V}|\sin(\theta)\,\underset{\sim}{j} \qquad \underset{\sim}{V} = |\underset{\sim}{V}|\cos(\theta)\,\underset{\sim}{i} + |\underset{\sim}{V}|\sin(\theta)\,\underset{\sim}{j}$$

Example 1:

Given: A force $\underset{\sim}{F}$ has a magnitude $|\underset{\sim}{F}| = 100$ (lbs) and an angle $\theta = 30$ (deg).

Find: Express the force $\underset{\sim}{F}$ in terms of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.



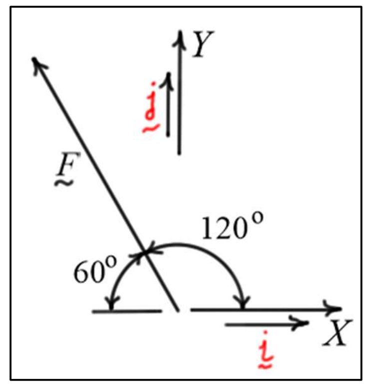
Solution:

$$\underset{\sim}{F} = 100\cos(30)\,\underset{\sim}{i} + 100\sin(30)\,\underset{\sim}{j} \approx 86.6\,\underset{\sim}{i} + 50\,\underset{\sim}{j} \text{ (lbs)}$$

Example 2:

Given: A force $\underset{\sim}{F}$ has a magnitude $|\underset{\sim}{F}| = 100$ (lbs) and an angle $\theta = 120$ (deg).

Find: Express the force $\underset{\sim}{F}$ in terms of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.



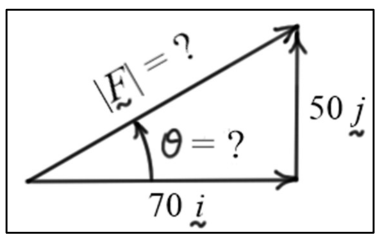
Solution:

$$\underset{\sim}{F} = 100\cos(120)\,\underset{\sim}{i} + 100\sin(120)\,\underset{\sim}{j} \approx -50\,\underset{\sim}{i} + 86.6\,\underset{\sim}{j} \text{ (lbs)}$$

Example 3:

Given: A force $\underset{\sim}{F} = 70\,\underset{\sim}{i} + 50\,\underset{\sim}{j}$ (lbs).

Find: The magnitude and direction of $\underset{\sim}{F}$.



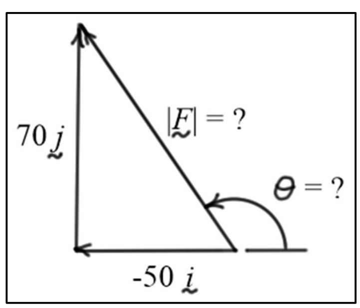
Solution:

$$|\underset{\sim}{F}| = \sqrt{70^2 + 50^2} \approx 86.0 \text{ (lbs)} \quad \text{and} \quad \theta = \tan^{-1}(50/70) \approx \begin{cases} 35.54 \text{ (deg)} \\ 0.6202 \text{ (rad)} \end{cases}$$

Example 4:

Given: A force $\underset{\sim}{F} = -50\,\underset{\sim}{i} + 70\,\underset{\sim}{j}$ (lbs).

Find: The magnitude and direction of $\underset{\sim}{F}$.

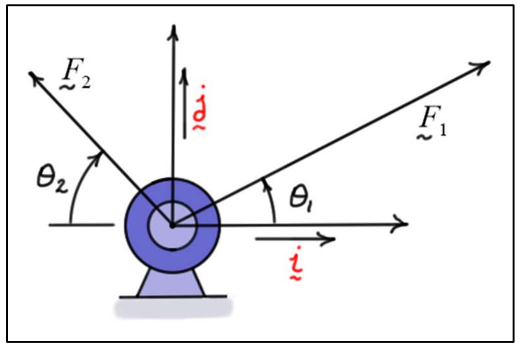


Solution:

$$|\underset{\sim}{F}| = \sqrt{(-50)^2 + 70^2} \approx 86.0 \text{ (lbs)} \qquad \theta = \tan^{-1}(70/-50) \approx \begin{cases} -54.46 + 180 = 125.5 \text{ (deg)} \\ -0.9505 + \pi = 2.191 \text{ (rad)} \end{cases}$$

Vector Addition

To ***add*** two or more vectors, simply express them in terms of the same unit vectors, and then add ***like components***.



Example 5:

Given: $\left|\underset{\sim}{F}_1\right| = 150 \text{ (lbs)}, \ \theta_1 = 20 \text{ (deg)}$

$\left|\underset{\sim}{F}_2\right| = 100 \text{ (lbs)}, \ \theta_2 = 60 \text{ (deg)}$

Find:

a) The ***total force*** $\underset{\sim}{F}$ acting on the support in terms of the unit vectors shown.

b) The ***magnitude*** and ***direction*** of $\underset{\sim}{F}$.

Solution:

a) The ***total force*** is the ***vector sum*** of the two forces.

$$\underset{\sim}{F}_1 = 150\cos(20)\,\underset{\sim}{i} + 150\sin(20)\,\underset{\sim}{j} \approx 140.95\,\underset{\sim}{i} + 51.3\,\underset{\sim}{j} \text{ (lbs)}$$

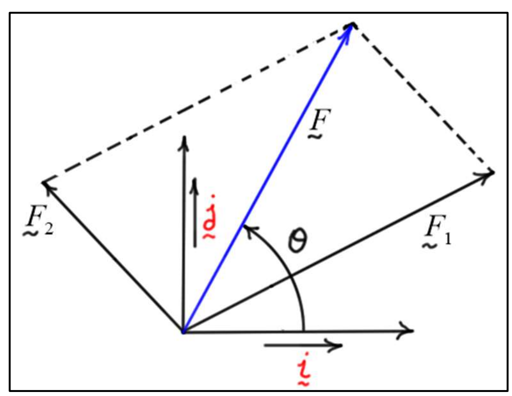
$$\underset{\sim}{F}_2 = -100\cos(60)\,\underset{\sim}{i} + 100\sin(60)\,\underset{\sim}{j} \approx -50\,\underset{\sim}{i} + 86.6\,\underset{\sim}{j} \text{ (lbs)}$$

$$\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 \approx (140.95 - 50)\,\underset{\sim}{i} + (51.3 + 86.6)\,\underset{\sim}{j} \approx 90.95\,\underset{\sim}{i} + 137.9\,\underset{\sim}{j} \text{ (lbs)}$$

b) $$\left|\underset{\sim}{F}\right| \approx \sqrt{90.95^2 + 137.9^2} \approx 165.2 \text{ (lbs)}$$

$$\theta \approx \tan^{-1}\left(137.9 / 90.95\right) \approx \begin{cases} 56.59 \text{ (deg)} \\ 0.9877 \text{ (rad)} \end{cases}$$

As you can see from the figure on the right, $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ form the sides of a parallelogram, and the sum $\underset{\sim}{F}$ forms the diagonal. The observation that vectors can be added geometrically in this way is called the ***parallelogram law of addition***. In general, the triangle formed by $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$, and $\underset{\sim}{F}$ is a non-right triangle. The lengths and angles within this triangle can be studied using the ***law*** of ***cosines*** and the ***law*** of ***sines*** as discussed in previous notes.

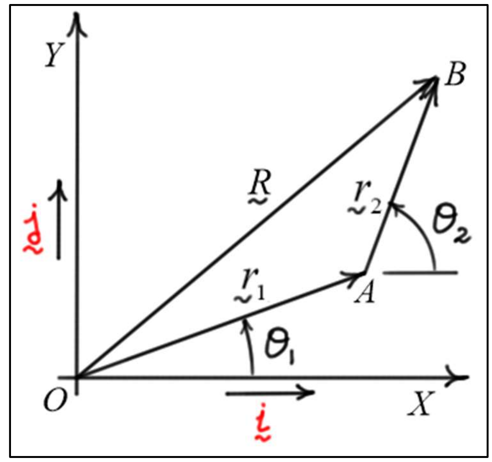


Example 6:

Given: The lengths and angles of a two link planar robot are $|\underset{\sim}{r}_1| = 3$ (ft), $|\underset{\sim}{r}_2| = 2$ (ft), $\theta_1 = 20$ (deg), and $\theta_2 = 70$ (deg).

Find: a) the position vector $\underset{\sim}{R}$ that defines the position of the endpoint of the robot relative to O.

b) the magnitude and direction of $\underset{\sim}{R}$.



Solution:

a) The position vector $\underset{\sim}{R}$ can be calculated by ***adding*** the vectors $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$.

$$\underset{\sim}{R} = \underset{\sim}{r}_1 + \underset{\sim}{r}_2 = \left(3\cos(20)\,\underset{\sim}{i} + 3\sin(20)\,\underset{\sim}{j}\right) + \left(2\cos(70)\,\underset{\sim}{i} + 2\sin(70)\,\underset{\sim}{j}\right)$$

$$\approx \left(2.8191\,\underset{\sim}{i} + 1.0261\,\underset{\sim}{j}\right) + \left(0.684\,\underset{\sim}{i} + 1.8794\,\underset{\sim}{j}\right)$$

$$\Rightarrow \boxed{\underset{\sim}{R} \approx 3.503\,\underset{\sim}{i} + 2.906\,\underset{\sim}{j} \text{ (ft)}}$$

b) $\boxed{|\underset{\sim}{R}| \approx \sqrt{3.503^2 + 2.906^2} \approx 4.55 \text{ (ft)}}$ and $\boxed{\theta \approx \tan^{-1}\left(2.906/3.503\right) \approx \begin{cases} 39.7 \text{ (deg)} \\ 0.6925 \text{ (rad)} \end{cases}}$

Scalar (Dot) Product

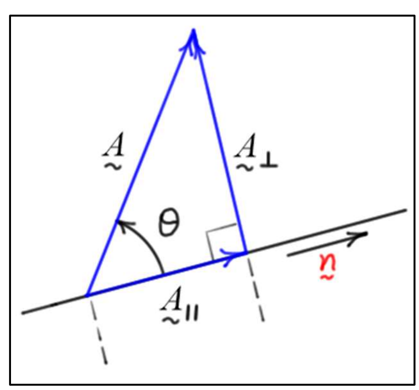
Geometric Definition

The ***scalar*** (or dot) product of two vectors is defined as follows.

$$\boxed{\underset{\sim}{A} \cdot \underset{\sim}{B} = |\underset{\sim}{A}|\,|\underset{\sim}{B}| \cos(\underset{\sim}{A}, \underset{\sim}{B})}$$

Here, $\cos(\underset{\sim}{A}, \underset{\sim}{B})$ represents the ***cosine*** of the angle between the tails of the two vectors. If one of the vectors is a ***unit vector***, then the scalar product is the ***projection*** of the vector in the direction of the unit vector.

$$\boxed{\underset{\sim}{A} \cdot \underset{\sim}{n} = |\underset{\sim}{A}|\,|\underset{\sim}{n}| \cos(\theta) = |\underset{\sim}{A}| \cos(\theta)}$$



The ***components*** of $\underset{\sim}{A}$ that are ***parallel*** and ***perpendicular*** to $\underset{\sim}{n}$ are

$\boxed{\underset{\sim}{A}_{\parallel} = \left(\underset{\sim}{A} \cdot \underset{\sim}{n}\right)\underset{\sim}{n}}$ and $\boxed{\underset{\sim}{A}_{\perp} = \underset{\sim}{A} - \underset{\sim}{A}_{\parallel}}$.

Calculation of the Dot Product of Two Vectors

Given two vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ expressed in terms of a pair of mutually perpendicular unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$, we ***calculate*** the ***dot product*** as follows.

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \left(a_x\,\underset{\sim}{i} + a_y\,\underset{\sim}{j}\right) \cdot \left(b_x\,\underset{\sim}{i} + b_y\,\underset{\sim}{j}\right) = a_x b_x + a_y b_y$$

The ***dot product*** of two vectors is ***zero*** if they are ***perpendicular*** to each other.

Example 7:

Given: Two vectors, $\underset{\sim}{A} = 10\,\underset{\sim}{i} + 2\,\underset{\sim}{j}$ and $\underset{\sim}{B} = 3\,\underset{\sim}{i} + 7\,\underset{\sim}{j}$

Find: The angle between the two vectors, θ.

Solution:

We can calculate the ***angle*** using the ***inverse cosine function***.

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}|\,|\underset{\sim}{B}|}\right) = \cos^{-1}\left(\frac{(10\times 3)+(2\times 7)}{\sqrt{10^2+2^2}\sqrt{3^2+7^2}}\right) \approx \cos^{-1}\left(\frac{44}{77.666}\right) \approx \begin{cases} 55.49 \text{ (deg)} \\ 0.9685 \text{ (rad)} \end{cases}$$

Example 8:

Given: A vector, $\underset{\sim}{A} = 2\,\underset{\sim}{i} + 8\,\underset{\sim}{j}$, and a unit vector $\underset{\sim}{n} = \frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$

Find: a) θ the ***angle*** between the two vectors

b) the component of vector $\underset{\sim}{A}$ ***parallel*** to unit vector $\underset{\sim}{n}$

c) the component of vector $\underset{\sim}{A}$ ***perpendicular*** to unit vector $\underset{\sim}{n}$

Solution:

a) We can calculate the angle using the inverse cosine function as before.

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{A} \cdot \underset{\sim}{n}}{|\underset{\sim}{A}|}\right) = \cos^{-1}\left(\frac{(2\times\frac{3}{5})+(8\times\frac{4}{5})}{\sqrt{2^2+8^2}}\right) \approx \cos^{-1}\left(\frac{7.6}{8.2462}\right) \approx \begin{cases} 22.83 \text{ (deg)} \\ 0.3985 \text{ (rad)} \end{cases}$$

b) The component of $\underset{\sim}{A}$ parallel to $\underset{\sim}{n}$: $\underset{\sim}{A}_{\parallel} = (\underset{\sim}{A} \cdot \underset{\sim}{n})\underset{\sim}{n} = 7.6\left(\frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}\right) = 4.56\,\underset{\sim}{i} + 6.08\,\underset{\sim}{j}$

c) The component of $\underset{\sim}{A}$ perpendicular to $\underset{\sim}{n}$:

$$\underset{\sim}{A}_{\perp} = \underset{\sim}{A} - \underset{\sim}{A}_{\parallel} \approx \left(2\,\underset{\sim}{i} + 8\,\underset{\sim}{j}\right) - \left(4.56\,\underset{\sim}{i} + 6.08\,\underset{\sim}{j}\right) \approx -2.56\,\underset{\sim}{i} + 1.92\,\underset{\sim}{j}$$

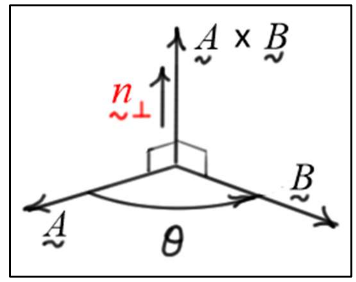
Check: $\underset{\sim}{A}_{\perp} \cdot \underset{\sim}{A}_{\parallel} \approx \left(-2.56\,\underset{\sim}{i} + 1.92\,\underset{\sim}{j}\right) \cdot \left(4.56\,\underset{\sim}{i} + 6.08\,\underset{\sim}{j}\right) \approx (-2.56 \times 4.56) + (1.92 \times 6.08) \approx 0$ ✓

Vector (Cross) Product

Geometric Definition

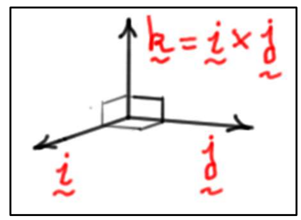
The ***cross*** product of two vectors is defines as follows.

$$\boxed{\underset{\sim}{A} \times \underset{\sim}{B} = \left(|\underset{\sim}{A}|\,|\underset{\sim}{B}| \sin(\underset{\sim}{A},\underset{\sim}{B}) \right) \underset{\sim}{n}_{\perp}}$$



Here, $\sin(\underset{\sim}{A},\underset{\sim}{B})$ is the ***sine*** of the angle between the tails of the two vectors, and $\underset{\sim}{n}_{\perp}$ is a unit vector ***perpendicular*** to the plane formed by vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$. The sense of $\underset{\sim}{n}_{\perp}$ is defined by the ***right-hand-rule***, that is, the right thumb points in the direction of $\underset{\sim}{n}_{\perp}$ when the fingers of the right hand point from $\underset{\sim}{A}$ to $\underset{\sim}{B}$.

Calculation



Given two vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ expressed in terms of mutually perpendicular unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$, we calculate the cross product as

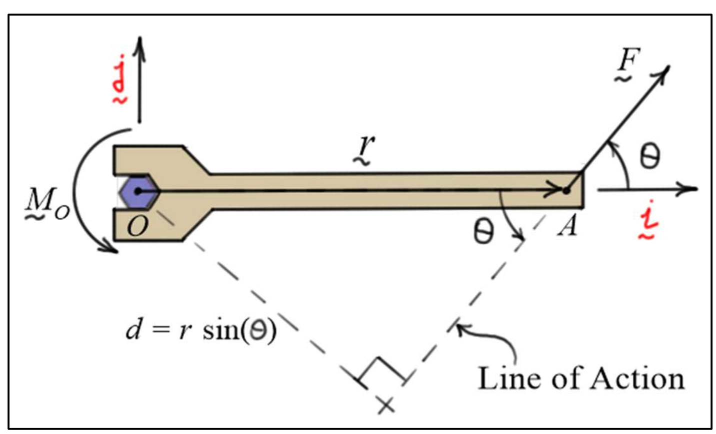
$$\boxed{\underset{\sim}{A} \times \underset{\sim}{B} = \left(a_x \underset{\sim}{i} + a_y \underset{\sim}{j} \right) \times \left(b_x \underset{\sim}{i} + b_y \underset{\sim}{j} \right) = \left(a_x b_y - a_y b_x \right) \underset{\sim}{k}}$$

The ***cross product*** of two vectors is ***zero*** if they are ***parallel*** to each other.

The above result can be calculated from the ***determinant form***. Since the cross product of two-dimensional vectors has no $\underset{\sim}{i}$ or $\underset{\sim}{j}$ components, we write

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix} = \left(a_x b_y - a_y b_x \right) \underset{\sim}{k}$$

Moment of a Force – Torque



The ***moment*** (or ***torque***) of a force about a point O is defined as the magnitude of the force $\left(|\underset{\sim}{F}| \right)$ multiplied by the ***perpendicular distance*** from the point to the ***line of action*** of the force $\left(d = r\sin(\theta) \right)$. The right-hand-rule defines the direction. So, it can be calculated using the ***cross product***.

$$\boxed{\underset{\sim}{M}_O = \underset{\sim}{r} \times \underset{\sim}{F}}$$

Here, $\underset{\sim}{r}$ is a position vector from O to the line of action of $\underset{\sim}{F}$.

Example 9:

Given: A force $\underset{\sim}{F} = 50\,\underset{\sim}{i} + 100\,\underset{\sim}{j}$ (lbs) is applied at a point A whose coordinates are $(3,2)$ (ft).

Find: a) $\underset{\sim}{M}_O$ the moment of the force about O the origin at $(0,0)$

b) d the perpendicular distance from O to the line of action of the force

Solution:

a) $$\boxed{\underset{\sim}{M}_O = \underset{\sim}{r} \times \underset{\sim}{F} = \left(3\,\underset{\sim}{i} + 2\,\underset{\sim}{j}\right) \times \left(50\,\underset{\sim}{i} + 100\,\underset{\sim}{j}\right) = \left((3 \times 100) - (2 \times 50)\right)\,\underset{\sim}{k} = 200\,\underset{\sim}{k} \text{ (ft-lbs)}}$$

b) $$\boxed{d = \frac{|\underset{\sim}{M}_O|}{|\underset{\sim}{F}|} = \frac{200}{\sqrt{50^2 + 100^2}} \approx \frac{200}{111.80} \approx 1.79 \text{ (ft)}}$$

Example 10:

Given: A force $\underset{\sim}{F} = 50\,\underset{\sim}{i} + 100\,\underset{\sim}{j}$ (lbs) is applied at a point A whose coordinates are $(3,-2)$ (ft).

Find: a) $\underset{\sim}{M}_B$ the moment of the force about point B whose coordinates are $(5,10)$ (ft)

b) d the perpendicular distance from B to the line of action of the force

Solution:

To calculate the moment using the cross product, we must first calculate the position vector that defines the position of A ***relative to*** B.

a) $$\boxed{\underset{\sim}{r} = \underset{\sim}{r}_{A/B} = \underset{\sim}{r}_A - \underset{\sim}{r}_B = \left(3\,\underset{\sim}{i} - 2\,\underset{\sim}{j}\right) - \left(5\,\underset{\sim}{i} + 10\,\underset{\sim}{j}\right) = -2\,\underset{\sim}{i} - 12\,\underset{\sim}{j} \text{ (ft)}}$$

$$\boxed{\underset{\sim}{M}_B = \underset{\sim}{r} \times \underset{\sim}{F} = \left(-2\,\underset{\sim}{i} - 12\,\underset{\sim}{j}\right) \times \left(50\,\underset{\sim}{i} + 100\,\underset{\sim}{j}\right) = \left((-2 \times 100) + (12 \times 50)\right)\underset{\sim}{k} = 400\,\underset{\sim}{k} \text{ (ft-lb)}}$$

b) $$\boxed{d = \frac{|\underset{\sim}{M}_B|}{|\underset{\sim}{F}|} = \frac{400}{\sqrt{50^2 + 100^2}} \approx \frac{400}{111.80} \approx 3.58 \text{ (ft)}}$$