Introductory Control Systems
Partial Fraction Expansions

- ***Linear ordinary differential equations*** (ODE's) can be solved using Laplace transforms. There are ***three steps*** in the solution process.
    1. The ***Laplace transform is used to convert*** a differential equation into an algebraic equation.
    2. The algebraic equation is manipulated to solve for the Laplace transformation of the solution. The concept of ***partial fractions*** is used (as necessary) to separate the solution into its elementary components.
    3. The ***inverse Laplace transform*** is taken of each of the elementary components to find the solution in the time domain.
- Given a ***single differential equation*** for the variable $x(t)$, application of the ***Laplace transform*** method gives an ***algebraic equation*** for $X(s)$, the Laplace transform of $x(t)$. This equation is then solved to find $X(s)$. In general, $X(s)$ will be a ***ratio*** of ***two polynomials*** in the complex variable s.
- If the form of this ratio is found in the Laplace transform table, then the ***inverse transform*** immediately provides the solution $x(t)$.
- However, if the form of $X(s)$ is ***more complicated*** than those found in the table, the concept of ***partial fractions*** can be used to ***separate*** the solution into its ***elementary components*** (all of which are found in the table). The ***sum*** of the ***inverse Laplace transforms*** of the elementary components provides the solution $x(t)$.
- As mentioned above, when the Laplace transform method is applied to a single differential equation, the Laplace transform of the solution is found in the form

$$X(s) = \frac{Q(s)}{P(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s}$$

- The ***form of the partial fraction expansion*** of $X(s)$ depends on the ***roots*** of the ***characteristic equation*** (denominator polynomial). Three cases are described below. In each case, the denominator polynomial is assumed to be of ***higher order*** than the numerator polynomial.

Case 1: Characteristic equation has *real, unequal* roots only

When the characteristic equation has N ***real, unequal*** roots, then $X(s)$ can be written as

$$X(s) = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_N)}$$

In this case, the ***partial fraction expansion*** can be written as the sum of N ***first-order*** systems. That is,

$$X(s) = \frac{K_1}{(s+s_1)} + \frac{K_2}{(s+s_2)} + \cdots + \frac{K_N}{(s+s_N)}$$

where the constants K_i $(i = 1, \ldots, N)$ are determined using the equation

$$K_i = \left[(s+s_i)X(s)\right]_{s=-s_i} = \frac{Q(-s_i)}{(-s_i+s_1)(-s_i+s_2)\cdots(-s_i+s_{i-1})(-s_i+s_{i+1})\cdots(-s_i+s_N)}$$

Case 2: Characteristic equation has *complex* roots only

When the characteristic equation has M ***pairs*** of ***complex-conjugate roots***, write $X(s)$ in the form

$$X(s) = \frac{Q(s)}{(s^2+2\zeta\omega_n s+\omega_n^2)_1(s^2+2\zeta\omega_n s+\omega_n^2)_2\cdots(s^2+2\zeta\omega_n s+\omega_n^2)_M}$$

In this case, the ***partial fraction expansion*** can be written as the sum of M second-order systems.

$$X(s) = \frac{A_1 s+B_1}{(s^2+2\zeta\omega_n s+\omega_n^2)_1} + \frac{A_2 s+B_2}{(s^2+2\zeta\omega_n s+\omega_n^2)_2} + \cdots + \frac{A_M s+B_M}{(s^2+2\zeta\omega_n s+\omega_n^2)_M}$$

The A_i and B_i $(i = 1, \ldots, M)$ can be determined by ***clearing fractions*** and solving the simultaneous equations that result from equating the coefficients of like powers of s.

Case 3: Characteristic equation has *real, unequal* and *complex* roots

When the characteristic equation has N ***real***, ***unequal roots*** and M ***pairs*** of ***complex-conjugate roots***, write $X(s)$ in the form

$$X(s) = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_N)(s^2+2\zeta\omega_n s+\omega_n^2)_1(s^2+2\zeta\omega_n s+\omega_n^2)_2\cdots(s^2+2\zeta\omega_n s+\omega_n^2)_M}$$

In this case, the ***partial fraction expansion*** can be written as the sum of N ***first-order*** systems and M ***second-order*** systems.

$$X(s)=\frac{K_1}{(s+s_1)}+\frac{K_2}{(s+s_2)}+\cdots+\frac{K_N}{(s+s_N)}+$$
$$\frac{A_1 s+B_1}{(s^2+2\zeta\omega_n s+\omega_n^2)_1}+\frac{A_2 s+B_2}{(s^2+2\zeta\omega_n s+\omega_n^2)_2}+\cdots+\frac{A_M s+B_M}{(s^2+2\zeta\omega_n s+\omega_n^2)_M}$$

The $K_i\ (i=1,\ldots N)$ can be determined as in Case 1, and once the K_i are known, the A_i and $B_i\ (i=1,\ldots,M)$ can be determined by ***clearing fractions*** and equating the coefficients of like powers of s as in Case 2.

Notes:

1. The method of ***clearing fractions*** can be used ***exclusively*** to solve for all the coefficients in the partial fraction expansions. However, the above method for determining the coefficients for the real roots is convenient in that they may be determined one at a time.
2. Partial fraction expansions can be defined for systems with ***repeated real roots*** as well, but they are not covered here.