

Elementary Engineering Mathematics

Application of Complex Numbers in Electric Circuits

Impedance in AC (Alternating Current) Circuits

In AC circuits, the steady-state voltages and currents are nearly **sinusoidal**. They alternate at some **frequency** ω (rad/sec) and have both **magnitude** and **phase**. We can analyze these signals using **complex numbers** and a **complex form** of **Ohm's law**. This form of Ohm's law relates the **sinusoidal signals** to the **impedances** of the circuit elements. The **impedances** are expressed as **complex numbers** and are measured in **ohms** (Ω). Impedance in an AC circuit is analogous to resistance in a DC (Direct Current) circuit.

Resistor

The impedance of a resistor is its resistance R ,
$$Z_R = R + j0 = Re^{j(0)} = R\angle 0^\circ$$
. The impedance of a resistor is the same at all frequencies (ω).

Capacitor

The impedance of a capacitor is
$$Z_C = -j/\omega C = (1/\omega C)e^{j(-90^\circ)} = (1/\omega C)\angle(-90^\circ)$$
. Here, ω is the **frequency** of the signals in (rad/s) and C is the **capacitance** measured in **farads** (f). Often C is provided in **micro-farads** (μf). A micro-farad is 10^{-6} farads.

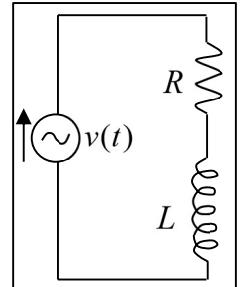
Inductor

The impedance of an inductor is
$$Z_L = j\omega L = (\omega L)e^{j(90^\circ)} = (\omega L)\angle(90^\circ)$$
. Here, L is the **inductance** measured in **henrys** (h). Often L is provided in **millihenrys** (mh). A millihenry is 10^{-3} henrys. As before, ω is the **frequency** of the signals in (rad/s).

Example #1

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RL series circuit with $R = 100$ (ohms) and $L = 100$ (mh). The impedance of the circuit is
$$Z = Z_R + Z_L$$
 and the frequency is
$$\omega = 120\pi \text{ (rad/s)}$$
.



Find: Find the complex impedance Z .

Solution:

$$Z = Z_R + Z_L = 100 + j(120\pi)(0.1) \Rightarrow Z \approx 100 + j37.7 \text{ (ohms)}$$

or

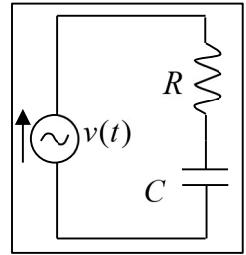
$$Z \approx 106.9 e^{j(20.7^\circ)} \approx 106.9 \angle(20.7^\circ) \text{ (ohms)}$$

Example #2

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RC series circuit with $R = 100$ (ohms) and $C = 20 \mu\text{f}$. The impedance of the circuit is $Z = Z_R + Z_C$ and the frequency is $\omega = 120\pi$ (rad/s).

Find: Find the complex impedance Z .



Solution:

$$Z = Z_R + Z_C = 100 - j/(120\pi)(20 \times 10^{-6}) \Rightarrow Z \approx 100 - j132.6 \text{ (ohms)}$$

or

$$Z \approx 166.1 e^{j(-52.98^\circ)} \approx 166.1 \angle(-52.98^\circ) \text{ (ohms)}$$

Example #3

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RLC series circuit with $R = 100$ (ohms), $L = 100$ (mh), and $C = 20 \mu\text{f}$. The impedance of the circuit is $Z = Z_R + Z_L + Z_C$ and the frequency is $\omega = 120\pi$ (rad/s).

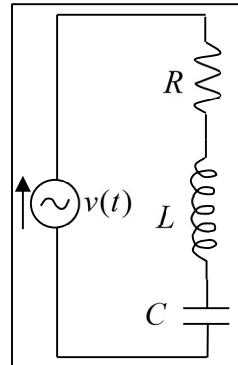
Find: Find the complex impedance Z .

Solution: (using the results from the examples above)

$$Z = Z_R + Z_L + Z_C \approx 100 + j(37.7 - 132.6) \Rightarrow Z \approx 100 - j94.9 \text{ (ohms)}$$

or

$$Z \approx 137.9 e^{j(-43.5^\circ)} \approx 137.9 \angle(-43.5^\circ) \text{ (ohms)}$$



Note: The **impedances** of **capacitors** and **inductors** are **functions of frequency**, whereas the impedance of a resistor is the same at all frequencies.

Complex Form of Ohm's Law

To find the currents in the above circuits, we use a complex form of Ohm's law which states that the **voltage drop across an impedance** is equal to the product of the complex current and complex impedance. That is, $V = IZ$. Rewriting this equation, we can find the current

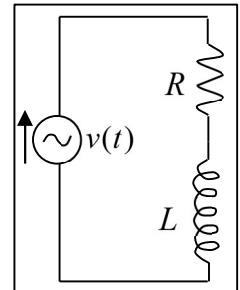
$$I = V/Z.$$

Example #4

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an *RL* series circuit with $R = 100$ (ohms) and $L = 100$ (mh). The impedance of the circuit is $Z = Z_R + Z_L$ and the frequency is $\omega = 120\pi$ (rad/s).

Find: Find the complex current I and the time-based current $i(t)$.



Solution:

From previous results, the impedance is $Z = Z_R + Z_L \approx 100 + j37.7$ (ohms)

or

$$Z \approx 106.9e^{j(20.7^\circ)} \approx 106.9 \angle (20.7^\circ) \text{ (ohms)}$$

The given voltage $v(t)$ can be written in complex form as $V = 110e^{j(0^\circ)} = 110 \angle (0^\circ)$.

So, the complex current is

$$I = V/Z \approx 110e^{j(0^\circ)} / 106.9e^{j(20.7^\circ)} \approx 1.029e^{j(-20.7^\circ)} \text{ (amps)}$$

and

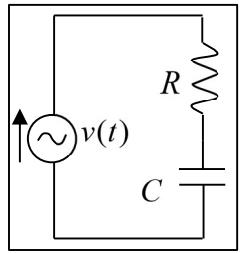
$$i(t) \approx 1.029 \cos(120\pi t - 0.3605) \text{ (amps)} \quad (\text{argument of cosine function} \sim \text{radians})$$

Note: We use only the **phase angle** of $v(t)$ and not " ωt " when we write the complex form. We must include the " ωt " when we write $i(t)$.

Example #5

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RC series circuit with $R = 100$ (ohms) and $C = 20$ (μf). The impedance of the circuit is $Z = Z_R + Z_C$ and the frequency is $\omega = 120\pi$ (rad/s).



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, the impedance is $Z = Z_R + Z_C \approx 100 - j132.6$ (ohms)

or

$$Z \approx 166.1e^{j(-52.98^\circ)} \approx 166.1\angle(-52.98^\circ) \text{ (ohms)}$$

So, the complex current is

$$I = V/Z \approx 110e^{j(0^\circ)} / 166.1e^{j(-52.98^\circ)} \approx 0.6623e^{j(52.98^\circ)} \text{ (amps)}$$

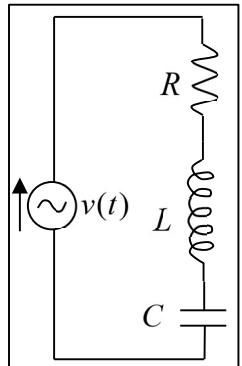
and

$$i(t) \approx 0.6623 \cos(120\pi t + 0.9246) \text{ (amps)} \quad (\text{argument of cosine function } \sim \text{radians})$$

Example #6

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RLC series circuit with $R = 100$ (ohms), $L = 100$ (mh), and $C = 20$ (μf). The impedance of the circuit is $Z = Z_R + Z_L + Z_C$ and the frequency is $\omega = 120\pi$ (rad/s).



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, we have $Z = Z_R + Z_L + Z_C \approx 100 - j94.9$ (ohms)

or

$$Z \approx 137.9e^{j(-43.5^\circ)} \approx 137.9\angle(-43.5^\circ) \text{ (ohms)}$$

So, the complex current is

$$I = V/Z \approx 110e^{j(0^\circ)} / 137.9e^{j(-43.5^\circ)} \approx 0.7977e^{j(43.5^\circ)} \text{ (amps)}$$

and

$$i(t) = 0.7977 \cos(120\pi t + 0.7592) \text{ (amps)} \quad (\text{argument of cosine function } \sim \text{radians})$$

The graph below shows a plot of three cycles of the voltage $v(t)$ and current $i(t)$ for the RLC circuit.

