Introductory Control Systems
Examples – Partial Fraction Expansions

1. Given: $X(s) = \dfrac{5s+3}{(s+1)(s+2)(s+3)}$

   Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

   Solution:

   Given the characteristic equation has ***real***, ***unequal roots***, the partial fraction expansion has the form

$$X(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

where the coefficients $K_i \ (i = 1,2,3)$ can be calculated as follows.

$$K_1 = \left[(s+1)X(s)\right]_{s=-1} = \left(\frac{5s+3}{(s+2)(s+3)}\right)_{s=-1} = -1$$

$$K_2 = \left[(s+2)X(s)\right]_{s=-2} = \left(\frac{5s+3}{(s+1)(s+3)}\right)_{s=-2} = 7$$

$$K_3 = \left[(s+3)X(s)\right]_{s=-3} = \left(\frac{5s+3}{(s+1)(s+2)}\right)_{s=-3} = -6$$

So, the partial fraction expansion is

$$\boxed{X(s) = \left(\frac{-1}{s+1}\right) + \left(\frac{7}{s+2}\right) + \left(\frac{-6}{s+3}\right)}$$

Using the Laplace transform table gives

$$\boxed{x(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}}$$

2. Given: $X(s)=\dfrac{5s+15}{(s+2)(s^2+3s+9)}$

Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

Solution:

Given the characteristic equation has ***one real root*** and a ***pair*** of ***complex conjugate*** roots, the partial fraction expansion is of the form

$$X(s)=\left(\frac{K}{s+2}\right)+\left(\frac{As+B}{s^2+3s+9}\right)$$

Here, the coefficient K can be found as before

$$K=\left[(s+2)X(s)\right]_{s=-2}=\left[\frac{5s+15}{s^2+3s+9}\right]_{s=-2}=\frac{5}{7}$$

and the coefficients A and B are found by ***clearing fractions*** as follows

$$\begin{aligned}5s+15&=K(s^2+3s+9)+(As+B)(s+2)\\&=(K+A)s^2+(2A+B+3K)s+(9K+2B)\end{aligned}$$

Equating the coefficients of the powers of s on both sides of the equation gives

$$\begin{aligned}&A+K=0 && (s^2)\\&2A+B+3K=5 && (s^1)\\&9K+2B=15 && (s^0)\end{aligned}$$

Solving the first and third equations gives $A=-\frac{5}{7}$ and $B=\frac{30}{7}$. So, the partial fraction expansion is

$$\boxed{X(s)=\frac{5}{7}\left(\frac{1}{s+2}\right)-\frac{5}{7}\left(\frac{s-6}{s^2+3s+9}\right)=\frac{5}{7}\left(\frac{1}{s+2}\right)-\frac{5}{7}\left(\frac{s+(-6)}{(s+\frac{3}{2})^2+\frac{27}{4}}\right)}$$

Using #4 and #18 in the Laplace transform tables gives

$$\boxed{x(t)=\frac{5}{7}\left[e^{-2t}-3.055e^{-1.5t}\sin(2.5981t+\phi)\right]}$$

$$\boxed{\phi=\tan^{-1}\left(\frac{2.5981}{-6-1.5}\right)=\begin{cases}-0.3335 & \text{(rad)}\\ & \text{or}\\ 2.8081 & \text{(rad)}\end{cases}}$$

The correct choice of ϕ must be made to ***satisfy*** the ***initial conditions*** of the differential equation (not given here).

3. <u>Given</u>: $X(s)=\dfrac{s^5+a_1s^4+a_2s^3+a_3s^2+a_4s+a_5}{s^6+11s^5+79s^4+427s^3+1510s^2+3800s+4000}$

<u>Find</u>: What is the form of $x(t)$? Identify the ***steady-state*** and ***transient*** parts of $x(t)$.

<u>Solution</u>:

Using the root solving feature on your calculator, the ***poles*** of $X(s)$ are found to be $\pm 5j$, $-2\pm 3.4641j$, -5, and -2. So, the partial fraction expansion and $x(t)$ have the forms

$$\boxed{X(s)=\left(\frac{K_1}{s+2}\right)+\left(\frac{K_2}{s+5}\right)+\left(\frac{A_1s+B_1}{s^2+25}\right)+\left(\frac{A_2s+B_2}{s^2+4s+16}\right)}$$

$$\boxed{x(t)=\underbrace{K_1e^{-2t}+K_2e^{-5t}+C_1e^{-2t}\sin\left(\sqrt{12}\,t+\phi_1\right)}_{\text{transient response}}+\underbrace{C_2\sin(5t+\phi_2)}_{\text{steady-state response}}}$$