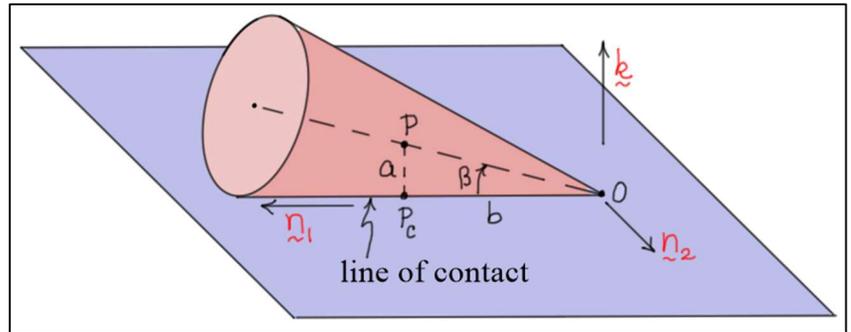**Intermediate Dynamics**
Rolling Constraints – Line Contact

Cone Rolling on a Flat Plane

In earlier notes, rolling (without slipping) with a single contact point was discussed. Consider now rolling when there is a *line of contact* between the bodies.

line of contact

For example, consider a *right circular cone* C rolling on a flat horizontal plane. If the plane is *fixed*, then all the points on the line of contact of the cone must have *zero velocity*, and the cone will roll in a *circular path* with the point O remaining fixed.

To analyze the kinematics of the cone as it rolls, the directions of a reference frame $S:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{k})$ are defined as shown in the diagram. The direction $\underset{\sim}{n}_1$ points along the *line of contact*, the direction $\underset{\sim}{k}$ is *normal* to the plane, and the direction $\underset{\sim}{n}_2 = \underset{\sim}{k} \times \underset{\sim}{n}_1$. Because all the points on the *contact line* have *zero velocity*, the *angular velocity* of the cone must be along the $\underset{\sim}{n}_1$ direction. That is,

$$\boxed{^{R}\underset{\sim}{\omega}_C = \omega\,\underset{\sim}{n}_1}$$

The *velocity* of any point of the cone can be calculated using the relative velocity equation. For example, suppose P_C is a *point* on the *contact line*, and P is a *point* along the cone's *centerline*, a distance a *above* P_C. Then, the *velocity* of P can be calculated as

$$^{R}\underset{\sim}{v}_P = {}^{R}\underset{\sim}{v}_{P_C} + {}^{R}\underset{\sim}{v}_{P/P_C} = \underbrace{{}^{R}\underset{\sim}{v}_{P_C}}_{\text{zero}} + \left({}^{R}\underset{\sim}{\omega}_C \times {}^{R}\underset{\sim}{r}_{P/P_C}\right) = \omega\,\underset{\sim}{n}_1 \times a\,\underset{\sim}{k}$$

$$\Rightarrow \boxed{^{R}\underset{\sim}{v}_P = -a\omega\,\underset{\sim}{n}_2}$$

Note that as points P are examined from the vertex of the cone to the center of its base the *lengths* a (and, hence, the *velocities* of the points P) *increase linearly*. Consequently, the cone rolls in a circular path with the point O remaining fixed.

The **angular acceleration** of the cone is found by **differentiating** $^R\underset{\sim}{\omega}_C$.

$$^R\underset{\sim}{\alpha}_C = \frac{^Rd}{dt}\left(\omega\underset{\sim}{n}_1\right) = \dot{\omega}\underset{\sim}{n}_1 + \omega\dot{\underset{\sim}{n}}_1 = \dot{\omega}\underset{\sim}{n}_1 + \omega\left(^R\underset{\sim}{\omega}_S \times \underset{\sim}{n}_1\right) = \dot{\omega}\underset{\sim}{n}_1 + \omega\left(\Omega\underset{\sim}{k}\times\underset{\sim}{n}_1\right) = \dot{\omega}\underset{\sim}{n}_1 + \omega\Omega\underset{\sim}{n}_2$$

Here, $^R\underset{\sim}{\omega}_S = \Omega\underset{\sim}{k}$ represents the angular velocity of the frame S in a fixed frame R. The angular rates ω and Ω are **not independent**, because the cone is **rolling** (without slipping).

To find a **relationship between** the **two rates**, consider again a point P on the centerline of the cone. As noted above, the velocity of P may be written as $^R\underset{\sim}{v}_P = -a\omega\underset{\sim}{n}_2$. However, it could also be written as $^R\underset{\sim}{v}_P = b\Omega\underset{\sim}{n}_2$ where b represents the distance from P_C to O. Comparing these two expressions yields the result, $\boxed{\Omega = -(a/b)\omega = -\omega\tan(\beta)}$, where β represents the **half angle** of the cone. Using this result, $^R\underset{\sim}{\alpha}_C$ may be written as

$$\boxed{^R\underset{\sim}{\alpha}_C = \dot{\omega}\underset{\sim}{n}_1 - \omega^2\tan(\beta)\underset{\sim}{n}_2}$$

The **acceleration** of P is found by differentiating the velocity $^R\underset{\sim}{v}_P$.

$$^R\underset{\sim}{a}_P = \frac{^Rd}{dt}\left(-a\omega\underset{\sim}{n}_2\right) = -a\left(\dot{\omega}\underset{\sim}{n}_2 + \omega\dot{\underset{\sim}{n}}_2\right) = -a\left(\dot{\omega}\underset{\sim}{n}_2 + \omega\left(\Omega\underset{\sim}{k}\times\underset{\sim}{n}_2\right)\right) = -a\left(\dot{\omega}\underset{\sim}{n}_2 - \omega\Omega\underset{\sim}{n}_1\right)$$

$$\Rightarrow \boxed{^R\underset{\sim}{a}_P = -a\dot{\omega}\underset{\sim}{n}_2 - a\omega^2\tan(\beta)\underset{\sim}{n}_1}$$

Beveled Gears

Beveled gears are a practical example of bodies that roll with a line of contact. Each gear may be thought of as part of a cone. Two contacting gears may be thought of as **two cones rolling on each other**. Consider, for example, the system of two beveled gears shown. Gear B is affixed to a shaft that rotates with speed Ω about the $\underset{\sim}{k}$ direction. Gear A rolls on gear B and rotates freely on the axle S. The axle rotates at a rate ω_z about the $\underset{\sim}{k}$ direction (pivoting about point O), and gear A rotates relative to the axle at a rate $\omega_{A/S}$.
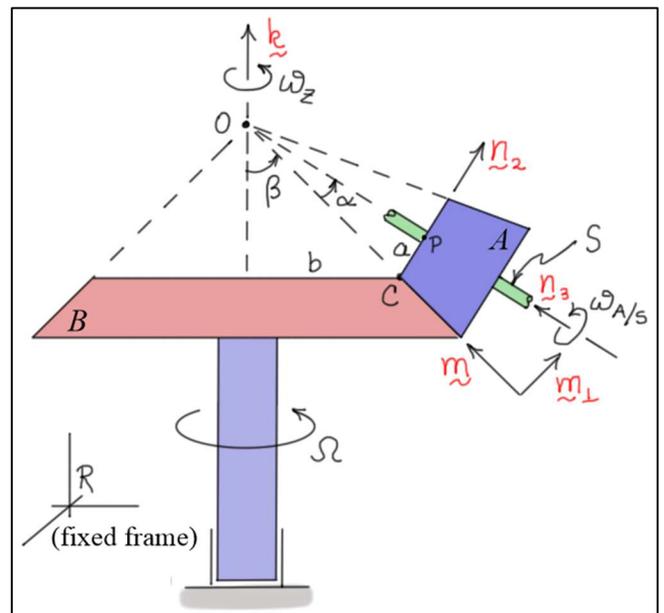
To analyze the kinematics of this system, define a unit vector set fixed in the axle as $S:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$. The direction $\underset{\sim}{n}_3$ is **pointed along** axle S towards O, the direction $\underset{\sim}{n}_2$ is **perpendicular** to the axle as shown, and the direction $\underset{\sim}{n}_1 = \underset{\sim}{n}_2 \times \underset{\sim}{n}_3$. The angles α and β are the **half angles** of the two cones (as shown), and the sum of these two angles is $\gamma \triangleq \alpha + \beta$. Point C is on the contact line between the two gears, and point P is a point on the axle.

To find a relationship between the angular rates Ω, ω_z, and $\omega_{A/S}$, use the concept of relative velocity. Assuming point C is the contact point on gear A, and point P is the center of that gear, write

$$^R\underset{\sim}{v}_C = {}^R\underset{\sim}{v}_P + {}^R\underset{\sim}{v}_{C/P}$$

Here,

$$^R\underset{\sim}{v}_P = -\left(b + aC_\gamma\right)\omega_z\, \underset{\sim}{n}_1$$

$$^R\underset{\sim}{v}_{C/P} = {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{C/P} = \left({}^R\underset{\sim}{\omega}_S + {}^S\underset{\sim}{\omega}_A\right)\times\left(-a\,\underset{\sim}{n}_2\right) = \left[\omega_z\,\underset{\sim}{k} + \omega_{A/S}\,\underset{\sim}{n}_3\right]\times\left(-a\,\underset{\sim}{n}_2\right)$$

$$= \left[\omega_z\left(S_\gamma\,\underset{\sim}{n}_2 + C_\gamma\,\underset{\sim}{n}_3\right) + \omega_{A/S}\,\underset{\sim}{n}_3\right]\times\left(-a\,\underset{\sim}{n}_2\right)$$

$$= a\left(\omega_{A/S} + \omega_z C_\gamma\right)\underset{\sim}{n}_1$$

Also, considering C to be the contact point on gear B, the velocity of C may be written as $^R\underset{\sim}{v}_C = -b\Omega\,\underset{\sim}{n}_1$. Setting the velocities of the two contact points equal to each other and simplifying gives

$$\boxed{\omega_{A/S} = (b/a)(\omega_z - \Omega)}$$

The Relative Angular Velocities $^B\underset{\sim}{\omega}_A$ and $^A\underset{\sim}{\omega}_B$

As noted above, when a cone rolls on a fixed plane, the **angular velocity** of the cone relative to the plane is **directed** along the **line of contact**. Similarly, when beveled gears roll on each other, the **relative angular velocities** $^B\underset{\sim}{\omega}_A$ and $^A\underset{\sim}{\omega}_B$ must be **directed** along the **line of contact**. Hence,

$$\boxed{^B\underset{\sim}{\omega}_A = \omega_{A/B}\underset{\sim}{m} = -{}^A\underset{\sim}{\omega}_B}$$

This result can be verified by finding the angular velocities and using the results found above. For example, using the angular velocity summation rule, write

$$^B\underset{\sim}{\omega}_A = {}^R\underset{\sim}{\omega}_A - {}^R\underset{\sim}{\omega}_B = \left(\omega_z \underset{\sim}{k} + \omega_{A/S}\underset{\sim}{n}_3\right) - \Omega\underset{\sim}{k} = \left(\omega_z - \Omega\right)\underset{\sim}{k} + \omega_{A/S}\underset{\sim}{n}_3$$

$$= \left(\omega_z - \Omega\right)\left(C_\beta \underset{\sim}{m} + S_\beta \underset{\sim}{m}_\perp\right) + \tfrac{b}{a}\left(\omega_z - \Omega\right)\left(C_\alpha \underset{\sim}{m} - S_\alpha \underset{\sim}{m}_\perp\right) \qquad (1)$$

$$\Rightarrow \boxed{{}^B\underset{\sim}{\omega}_A = \left(\omega_z - \Omega\right)\left(\left(C_\beta + \tfrac{b}{a}C_\alpha\right)\underset{\sim}{m} + \left(S_\beta - \tfrac{b}{a}S_\alpha\right)\underset{\sim}{m}_\perp\right)}$$

Here,

$$\frac{b}{a} = \frac{b}{\left|\overline{OC}\right|} \cdot \frac{\left|\overline{OC}\right|}{a} = \frac{S_\beta}{S_\alpha}$$

$$C_\beta + \tfrac{b}{a}C_\alpha = C_\beta + \frac{S_\beta C_\alpha}{S_\alpha} = \frac{C_\beta S_\alpha + S_\beta C_\alpha}{S_\alpha} = \frac{S_{\alpha+\beta}}{S_\alpha}$$

$$S_\beta - \tfrac{b}{a}S_\alpha = S_\beta - \frac{S_\beta S_\alpha}{S_\alpha} = \frac{S_\beta S_\alpha - S_\beta S_\alpha}{S_\alpha} = 0$$

Substituting these results into Eq. (1) gives

$$\boxed{{}^B\underset{\sim}{\omega}_A = \left(\omega_z - \Omega\right)\left(\frac{S_{\alpha+\beta}}{S_\alpha}\right)\underset{\sim}{m}}$$

So, the relative angular velocity is directed along the line of contact as expected.