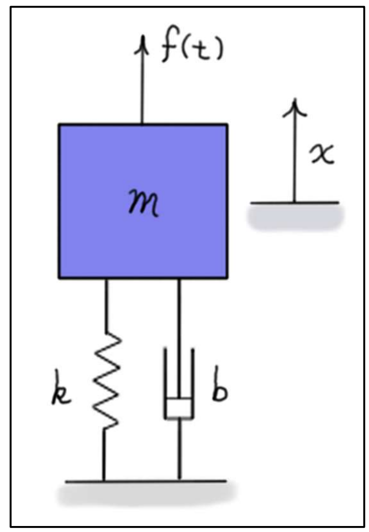
Introductory Control Systems
Transfer Functions

Single-Input, Single-Output (SISO) Systems

- For ***linear*** systems that have a single input and a single output, a single ***transfer function*** can be defined that quantifies the dynamic behavior of the system.
- Mathematically, the ***transfer function*** is defined as the ***Laplace transform of the output divided by the Laplace transform of the input***, assuming all ***initial values*** are ***zero***.
- As an example, consider the single degree-of-freedom mass, spring, damper system shown with a forcing function $f(t)$. For this system, the ***differential equation of motion*** is

$$\boxed{m\ddot{x} + b\dot{x} + kx = f(t)}$$



- ***Applying Laplace transforms*** to both sides of the equation ***assuming all initial values are zero*** gives

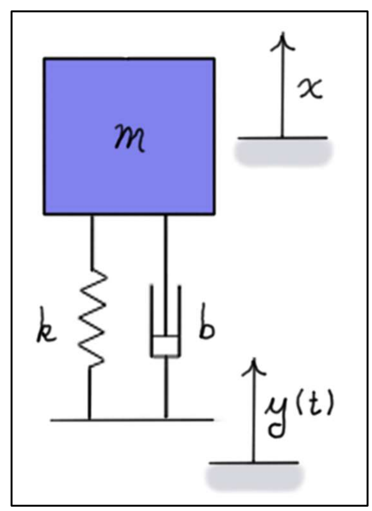
$$(ms^2 + bs + k)X(s) = F(s)$$

Given that the ***input*** of the system is $f(t)$ and the ***output*** is $x(t)$, the system transfer function is defined to be

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}}$$

- For comparison, consider the same system with the ***base motion*** $y(t)$ as the system input. The differential equation of motion is

$$\boxed{m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky}$$



- Applying Laplace transforms to this equation gives

$$(ms^2 + bs + k)X(s) = (bs + k)Y(s)$$

Given the ***input*** of the system is $y(t)$ and the ***output*** is $x(t)$, the ***system transfer function*** is

$$\boxed{\frac{X(s)}{Y(s)} = \frac{bs + k}{ms^2 + bs + k}}$$

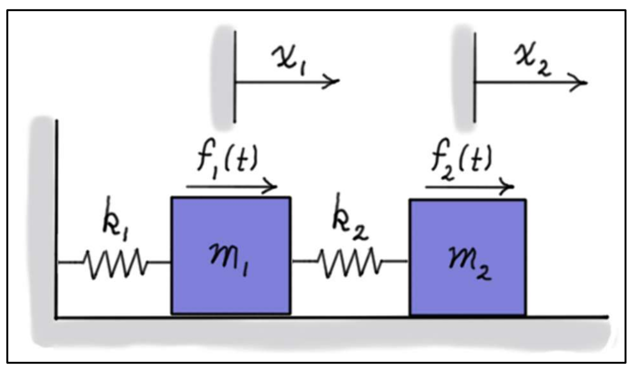
- If the input to the system is the ***impulse function***, $\delta(t)$, then $X(s)$ the Laplace transform of the response is equal the transfer function, since $F(s) = \mathcal{L}(\delta(t)) = 1$. In this case, the transfer function describes the ***impulse response*** of the system.
- In general, a transfer function will be a ratio of two polynomials.

$$\frac{X(s)}{F(s)} = \frac{s^m + b_m s^{m-1} + b_{m-1} s^{m-2} + \cdots + b_1}{s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_1}$$

- The ***roots*** of the ***numerator*** are called the ***zeros*** of the system, and the ***roots*** of the ***denominator*** are called the ***poles*** of the system. In these notes, the order of the numerator is assumed to be ***less than*** the order of the denominator, that is, $m < n$.

Multiple-Input, Multiple-Output (MIMO) Systems

- For ***linear*** systems with ***M input variables*** and ***N output variables***, we define $M \times N$ transfer functions, one relating ***each input/output pair***. Together, these transfer functions can be used to quantify the behavior of the system. As before, the transfer function is defined as the Laplace transform of the output variable divided by the Laplace transform of the input variable, assuming all initial values are zero.
- As an ***example***, consider the two degree-of-freedom, mass-spring system shown with forcing functions $f_1(t)$ and $f_2(t)$. It can be shown that the equations of motion for this system can be written as



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t)$$
$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = f_2(t)$$

- Applying Laplace transforms to these equations assuming all initial values are zero and writing the resulting equations in ***matrix form*** gives

$$\begin{bmatrix} m_1 s^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = [A]\{X(s)\} = \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} = \{F(s)\}$$

- From this equation, ***four transfer functions*** can be defined. Note that for transfer functions involving $F_1(s)$, $F_2(s)$ is taken as zero, and for the transfer functions involving $F_2(s)$, $F_1(s)$ is taken as zero. The four transfer functions are

  $$\frac{X_1(s)}{F_1(s)}, \frac{X_1(s)}{F_2(s)}, \frac{X_2(s)}{F_1(s)}, \text{ and } \frac{X_2(s)}{F_2(s)}$$

- ***Using Cramer's Rule, the boxed equation*** above can be solved for $X_1(s)$ and $X_2(s)$.

  $$X_1(s) = \frac{\det\begin{pmatrix} F_1(s) & -k_2 \\ F_2(s) & m_2 s^2 + k_2 \end{pmatrix}}{\det[A]} = \left(\frac{m_2 s^2 + k_2}{\det[A]}\right) F_1(s) + \left(\frac{k_2}{\det[A]}\right) F_2(s)$$

  and

  $$X_2(s) = \frac{\det\begin{pmatrix} m_1 s^2 + k_1 + k_2 & F_1(s) \\ -k_2 & F_2(s) \end{pmatrix}}{\det[A]} = \left(\frac{k_2}{\det[A]}\right) F_1(s) + \left(\frac{m_1 s^2 + k_1 + k_2}{\det[A]}\right) F_2(s)$$

  where $\boxed{\det[A] = (m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}$.

- From these results, the ***four transfer functions*** of the system are

  $$\boxed{\frac{X_1(s)}{F_1(s)} = \left(\frac{m_2 s^2 + k_2}{\det[A]}\right)} \quad \boxed{\frac{X_1(s)}{F_2(s)} = \left(\frac{k_2}{\det[A]}\right)} \quad \boxed{\frac{X_2(s)}{F_1(s)} = \left(\frac{k_2}{\det[A]}\right)} \quad \boxed{\frac{X_2(s)}{F_2(s)} = \left(\frac{m_1 s^2 + k_1 + k_2}{\det[A]}\right)}$$

  Note that all four transfer functions have the ***same characteristic equation***.

Experimental Determination of Transfer Functions

- To ***measure*** transfer functions ***experimentally***, ***actuators*** are used to excite the system, ***sensors*** to measure the system excitation (input) and response (output), and a ***data acquisition system*** to record the signals.
- MATLAB's ***system identification toolbox*** uses these signals to ***estimate*** the transfer function.
- A ***dynamic signal analyzer*** can also be used. In addition to recording the system input and output signals, it can calculate their ***Fast Fourier Transforms*** (FFT's) and display the ratio (output/input) in the form of a Bode diagram. The Bode diagram is one way of ***graphically displaying*** a transfer function. It displays the ***frequency response*** of the system. Bode diagrams are presented in later notes.