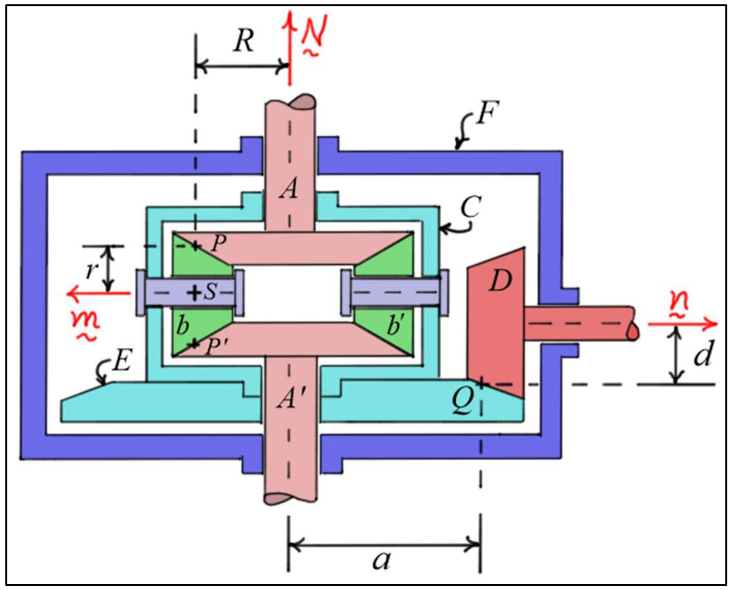
Intermediate Dynamics
Example: Differential Gear Set
(Reference: Kane and Levinson, *Dynamics: Theory and Applications*, McGraw-Hill, 1985)

Nomenclature

F : outer casing
(all inner components rotate relative to it)
D : drive shaft and rigidly attached bevel gear
E : beveled gear rigidly attached to casing C
C : inner casing with rigidly attached pins
b : beveled gear
(spins on pin rigidly attached to C)
b' : beveled gear
(spins on pin rigidly attached to C)
A : left shaft and rigidly attached bevel gear
(rotates relative to casings C and F)
A' : right shaft and rigidly attached bevel gear
(rotates relative to casings C and F)



Problem: The angular velocities of the drive shaft and the left and right wheel shafts relative to the outer casing F can be written as follows.

$$\boxed{{}^F\underset{\sim}{\omega}_D = \omega\,\underset{\sim}{n}} \qquad \boxed{{}^F\underset{\sim}{\omega}_A = \Omega\,\underset{\sim}{N}} \qquad \boxed{{}^F\underset{\sim}{\omega}_{A'} = \Omega'\,\underset{\sim}{N}}$$

Show that the three angular rates satisfy the following equation.

$$\boxed{\Omega + \Omega' = 2(d/a)\omega}$$

Solution:

The points P and P' represent points in contact between the bevel gear b and the bevel gears attached to the left and right wheel shafts. The velocity of point P relative to the inner casing C can be written in terms of the angular velocities of the bevel gears A and b relative to C as follows.

$${}^C\underset{\sim}{v}_P = {}^C\underset{\sim}{\omega}_A \times R\,\underset{\sim}{m} = R\,\omega_{A/C}\left(\underset{\sim}{N} \times \underset{\sim}{m}\right) \qquad \text{and} \qquad {}^C\underset{\sim}{v}_P = {}^C\underset{\sim}{\omega}_b \times r\,\underset{\sim}{N} = -r\,\omega_{b/C}\left(\underset{\sim}{N} \times \underset{\sim}{m}\right)$$

Comparing these results gives the following relationship between the angular velocities of A and b relative to C.

$$\boxed{\omega_{A/C} = -(r/R)\,\omega_{b/C}} \qquad (1)$$

The velocity of point P' relative to the inner casing C can be written in terms of the angular velocities of the bevel gears A and b relative to C as follows.

$${}^{C}\underset{\sim}{v}_{P'} = {}^{C}\underset{\sim}{\omega}_{A'} \times R\,\underset{\sim}{m} = R\,\omega_{A'/C}\left(\underset{\sim}{N} \times \underset{\sim}{m}\right) \qquad \text{and} \qquad {}^{C}\underset{\sim}{v}_{P'} = {}^{C}\underset{\sim}{\omega}_{b} \times \left(-r\,\underset{\sim}{N}\right) = r\,\omega_{b/C}\left(\underset{\sim}{N} \times \underset{\sim}{m}\right)$$

Comparing these results gives the following relationship between the angular velocities of A' and b relative to C.

$$\boxed{\omega_{A'/C} = \left(r/R\right)\omega_{b/C}} \qquad (2)$$

Comparing Eqs. (1) and (2) above, it is clear that

$$\boxed{\omega_{A/C} = -\omega_{A'/C}} \qquad (3)$$

The angular velocity of the drive shaft relative to the outer casing F can be related to the angular velocity of the inner casing C relative to F by calculating the velocity of point Q relative to F. Because Q is the contact point between D and E (which is rigidly attached to C), the velocity of Q relative to the outer casing F can be written as follows.

$${}^{F}\underset{\sim}{v}_{Q} = {}^{F}\underset{\sim}{\omega}_{D} \times -d\,\underset{\sim}{N} = d\,\omega\left(\underset{\sim}{N} \times \underset{\sim}{n}\right) \qquad \text{and} \qquad {}^{F}\underset{\sim}{v}_{Q} = {}^{F}\underset{\sim}{\omega}_{C} \times a\,\underset{\sim}{n} = a\,\omega_{C/F}\left(\underset{\sim}{N} \times \underset{\sim}{n}\right)$$

Also, using the summation rule for angular velocities, ${}^{F}\underset{\sim}{\omega}_{C}$ the angular velocity of inner casing C relative to the outer casing F can be written as follows.

$${}^{F}\underset{\sim}{\omega}_{C} = {}^{F}\underset{\sim}{\omega}_{A} + {}^{A}\underset{\sim}{\omega}_{C} = \Omega\,\underset{\sim}{N} + \omega_{C/A}\underset{\sim}{N} = \left(\Omega + \omega_{C/A}\right)\underset{\sim}{N} = \omega_{C/F}\,\underset{\sim}{N}$$

Comparing the last three equations and the fact that $\boxed{{}^{A}\underset{\sim}{\omega}_{C} = -{}^{C}\underset{\sim}{\omega}_{A}}$ gives the following result relating the angular velocity of the drive shaft to the angular velocities of the left shaft A and the angular velocity of A relative to the inner casing.

$$\boxed{d\,\omega = a\,\omega_{C/F} = a\left(\Omega + \omega_{C/A}\right) = a\left(\Omega - \omega_{A/C}\right)} \qquad (4)$$

Alternatively, ${}^{F}\underset{\sim}{\omega}_{C}$ the angular velocity of inner casing C relative to the outer casing F can be written as follows.

$${}^{F}\underset{\sim}{\omega}_{C} = {}^{F}\underset{\sim}{\omega}_{A'} + {}^{A'}\underset{\sim}{\omega}_{C} = \Omega'\,\underset{\sim}{N} + \omega_{C/A'}\underset{\sim}{N} = \left(\Omega' + \omega_{C/A'}\right)\underset{\sim}{N} = \omega_{C/F}\,\underset{\sim}{N}$$

Using this result with the equations for ${}^F\underset{\sim}{v}_Q$ and the fact that $\boxed{{}^{A'}\underset{\sim}{\omega}_C = -{}^C\underset{\sim}{\omega}_{A'}}$ gives the following result relating the angular velocity of the drive shaft to the angular velocities of the right shaft A' and the angular velocity of A' relative to the inner casing.

$$\boxed{d\omega = a\,\omega_{C/F} = a\left(\Omega' + \omega_{C/A'}\right) = a\left(\Omega' - \omega_{A'/C}\right)} \tag{5}$$

Adding Eqs. (4) and (5) and using Eq. (3) gives the desired result.

$$2\,d\,\omega = a\left(\Omega - \omega_{A/C}\right) + a\left(\Omega' - \omega_{A'/C}\right) = a\left(\Omega - \cancel{\omega_{A/C}}\right) + a\left(\Omega' + \cancel{\omega_{A/C}}\right) = a\left(\Omega + \Omega'\right)$$

$$\Rightarrow \boxed{\Omega + \Omega' = 2\left(d/a\right)\omega}$$

These results can be used to gain some insight into the motion of the internal components of the gear set. For example, consider the following two cases.

Case 1: $\Omega = \Omega'$ (… car going straight)

$$\boxed{\Omega = \Omega' = \left(d/a\right)\omega} \qquad \boxed{\omega_{A/C} = \Omega - \left(d\,/\,a\right)\omega \equiv 0 = \omega_{A'/C}} \qquad \boxed{\omega_{b/C} = 0}$$

So, when the car is going straight, the inner casing C rotates with the wheel shafts and the bevel gears b and b' remain fixed relative to C.

Case 2: $\Omega' = \left(3/4\right)\Omega$ (… car in a right turn)

$$\boxed{\Omega = \left(8d/7a\right)\omega} \quad \boxed{\Omega' = \left(6d/7a\right)\omega} \quad \boxed{\omega_{A/C} = \left(d/7a\right)\omega = -\omega_{A'/C}} \quad \boxed{\omega_{b/C} = -\left(Rd/7ra\right)\omega}$$

In a turn, the casing C rotates at a different rate than either of the wheel shafts, and the bevel gears b and b' rotate relative to C.

As a check on the above results, the angular velocity of bevel gear b relative to the left shaft A for Case 2 can be written using the summation rule (for angular velocities) as follows.

$$\boxed{{}^A\underset{\sim}{\omega}_b = {}^C\underset{\sim}{\omega}_b + {}^A\underset{\sim}{\omega}_C = -\left(Rd/7ra\right)\omega\,\underset{\sim}{m} - \left(d/7a\right)\omega\,\underset{\sim}{N} = -\left(d/7ar\right)\omega\left[R\underset{\sim}{m} + r\underset{\sim}{N}\right]}$$

Note that ${}^A\underset{\sim}{\omega}_b$ is directed along the tangent line of gears b and A as it should be.