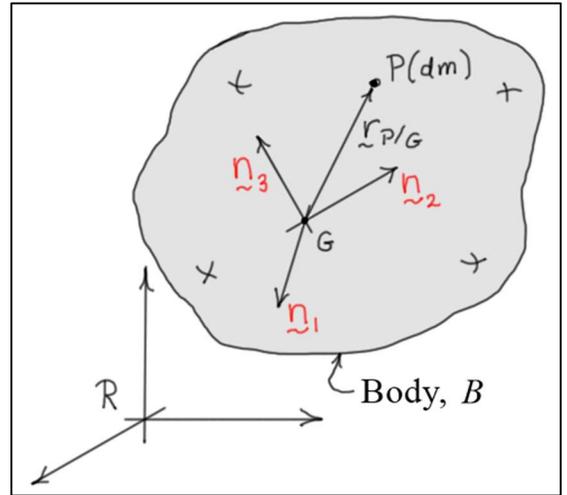


Intermediate Dynamics

Angular Momentum of a Rigid Body about its Mass Center

To calculate the *angular momentum* of a rigid body about its mass center, consider the body B shown in the figure. The point P represents *any point* in the body, dm represents the elemental mass of the body associated with P , and $\underline{r}_{P/G}$ represents the position vector of P with respect to the mass-center G . The *angular momentum* of B about G is defined as follows



$$\underline{H}_G = \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_P) dm$$

Using the kinematic formula for *two points fixed on a rigid body*, and the *definition of center of mass* (i.e. $\int_B \underline{r}_{P/G} dm = 0$), this expression can be rewritten as follows

$$\begin{aligned} \underline{H}_G &= \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_P) dm = \int_B (\underline{r}_{P/G} \times ({}^R \underline{v}_G + {}^R \underline{v}_{P/G})) dm \\ &= \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_G) dm + \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_{P/G}) dm \\ &= \left(\int_B \underline{r}_{P/G} dm \right) \times {}^R \underline{v}_G + \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_{P/G}) dm \\ &= \int_B (\underline{r}_{P/G} \times ({}^R \underline{\omega}_B \times \underline{r}_{P/G})) dm \end{aligned}$$

Using the *vector identity* $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$, the result above may be rewritten as

$$\underline{H}_G = \int_B (\underline{r}_{P/G} \times ({}^R \underline{\omega}_B \times \underline{r}_{P/G})) dm = \int_B (\underline{r}_{P/G} \cdot \underline{r}_{P/G}) {}^R \underline{\omega}_B dm - \int_B (\underline{r}_{P/G} \cdot {}^R \underline{\omega}_B) \underline{r}_{P/G} dm$$

Setting $\underline{r}_{P/G} = x \underline{n}_1 + y \underline{n}_2 + z \underline{n}_3$ and ${}^R \underline{\omega}_B = \omega_1 \underline{n}_1 + \omega_2 \underline{n}_2 + \omega_3 \underline{n}_3$, the above equation can be expanded as follows.

$$\begin{aligned}
\mathbf{H}_G &= \int_B (x^2 + y^2 + z^2) (\omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3) dm - \\
&\quad \int_B (x\omega_1 + y\omega_2 + z\omega_3) (x\mathbf{n}_1 + y\mathbf{n}_2 + z\mathbf{n}_3) dm \\
&= \int_B r^2 (\omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3) dm - \int_B (x\omega_1 + y\omega_2 + z\omega_3) (x\mathbf{n}_1 + y\mathbf{n}_2 + z\mathbf{n}_3) dm \\
&= \int_B (r^2 \omega_1 - x(x\omega_1 + y\omega_2 + z\omega_3)) \mathbf{n}_1 dm + \\
&\quad \int_B (r^2 \omega_2 - y(x\omega_1 + y\omega_2 + z\omega_3)) \mathbf{n}_2 dm + \\
&\quad \int_B (r^2 \omega_3 - z(x\omega_1 + y\omega_2 + z\omega_3)) \mathbf{n}_3 dm \\
&= \int_B ((y^2 + z^2)\omega_1 - xy\omega_2 - xz\omega_3) \mathbf{n}_1 dm + \\
&\quad \int_B (-xy\omega_1 + (x^2 + z^2)\omega_2 - yz\omega_3) \mathbf{n}_2 dm + \\
&\quad \int_B (-xz\omega_1 - yz\omega_2 + (x^2 + y^2)\omega_3) \mathbf{n}_3 dm
\end{aligned}$$

Now, since the integrals do not depend on the angular velocity components (ω_1 , ω_2 , and ω_3), the above equation can be rewritten as

$$\begin{aligned}
\mathbf{H}_G &= \left(\omega_1 \int_B (y^2 + z^2) dm + \omega_2 \int_B (-xy) dm + \omega_3 \int_B (-xz) dm \right) \mathbf{n}_1 + \\
&\quad \left(\omega_1 \int_B (-xy) dm + \omega_2 \int_B (x^2 + z^2) dm + \omega_3 \int_B (-yz) dm \right) \mathbf{n}_2 + \\
&\quad \left(\omega_1 \int_B (-xz) dm + \omega_2 \int_B (-yz) dm + \omega_3 \int_B (x^2 + y^2) dm \right) \mathbf{n}_3
\end{aligned}$$

or

$$\boxed{\mathbf{H}_G = (I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3) \mathbf{n}_1 + (-I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3) \mathbf{n}_2 + (-I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3) \mathbf{n}_3}$$

Here, the *integrals* are now recognized as the *moments* and *products of inertia* of the body about axes parallel to $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ and passing through the mass center G .

Representation of Angular Momentum as a Matrix-Vector Product

The above result for the angular momentum vector \underline{H}_G is easier to remember when the following *matrix-vector product* is used.

$$\begin{Bmatrix} \underline{H}_G \cdot \underline{n}_1 \\ \underline{H}_G \cdot \underline{n}_2 \\ \underline{H}_G \cdot \underline{n}_3 \end{Bmatrix} = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

Representation of Angular Momentum as a Dyadic-Vector Product

The angular momentum vector \underline{H}_G can also be written as the “dot” product of the inertia dyadic with the angular velocity vector. That is,

$$\underline{H}_G = \underline{I}_G \cdot \underline{\omega}$$

This is easily shown by substituting for \underline{I}_G and $\underline{\omega}$ in this expression and expanding.

$$\begin{aligned} \underline{H}_G &= \left(\sum_{i=1}^3 \sum_{j=1}^3 I_{ij}^G \underline{n}_i \underline{n}_j \right) \cdot \left(\sum_{k=1}^3 \omega_k \underline{n}_k \right) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \omega_k I_{ij}^G \underline{n}_i (\underline{n}_j \cdot \underline{n}_k) = \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \omega_k I_{ij}^G \underline{n}_i \delta_{jk} = \sum_{i=1}^3 \sum_{j=1}^3 I_{ij}^G \omega_j \underline{n}_i = \\ &= \left(I_{11}^G \omega_1 + I_{12}^G \omega_2 + I_{13}^G \omega_3 \right) \underline{n}_1 + \left(I_{21}^G \omega_1 + I_{22}^G \omega_2 + I_{23}^G \omega_3 \right) \underline{n}_2 + \left(I_{31}^G \omega_1 + I_{32}^G \omega_2 + I_{33}^G \omega_3 \right) \underline{n}_3 \\ &= \left(I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3 \right) \underline{n}_1 + \left(-I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3 \right) \underline{n}_2 + \left(-I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3 \right) \underline{n}_3 \end{aligned}$$

In the above expression, δ_{jk} is the **Kronecker delta symbol** and is equal to **one** if $j = k$ and **zero** if $j \neq k$. This result shows that a *dyadic-vector product* can be represented using *matrix multiplication*.

Question: When is \underline{H}_G parallel to $\underline{\omega}$? In two dimensions? In three dimensions?