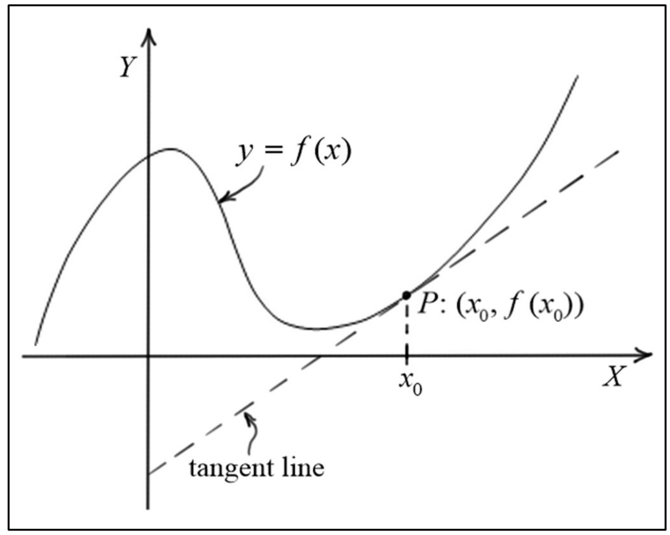
Elementary Engineering Mathematics
Introduction to the Derivative of a Function

Consider a continuous function of a single variable $y = f(x)$. The ***derivative*** of the function at any point P is simply the ***slope*** of the function at that point. The line that has the same slope as $f(x)$ and passes through P is called the ***tangent line*** at that point.



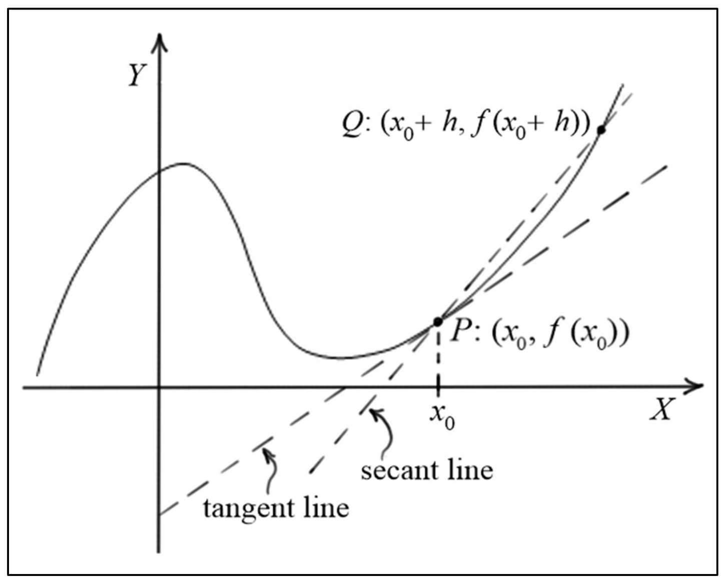
We can find the slope of the tangent line using a ***limiting process*** as follows. First, define a ***secant line*** that passes through points $P(x_0, f(x_0))$ and $Q(x_0 + h, f(x_0 + h))$. Then, the slope of the secant line is

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}} \quad \text{(secant line)}$$

The slope of the tangent line is

$$\boxed{\lim_{h \to 0}\left(\frac{f(x_0 + h) - f(x_0)}{h}\right)} \quad \text{(tangent line)}$$

In the limit as Q moves to P, the slope of the secant line becomes the same as the slope of the tangent line.



If this limit exists, the function is said to be ***differentiable*** at x_0, and the limit itself is called the ***derivative*** of f at x_0. It is common to denote the derivative as

$$\boxed{\lim_{h \to 0}\left(\frac{f(x_0 + h) - f(x_0)}{h}\right) = \left.\frac{df}{dx}\right|_{x = x_0} = f'(x_0)} \tag{1}$$

Once the slope is found, the ***equation*** of the ***tangent line*** can be found using the ***point-slope form*** for the equation of a line. The tangent line can be used as an ***approximation*** to the function $f(x)$ near x_0.

<u>Example 1</u>:

<u>Given</u>: In previous notes, we found that the path of a golf ball (neglecting air resistance) was defined by the function

$$\boxed{y = f(x) \approx 73.54\left(\frac{x}{61.71}\right) - 16.1\left(\frac{x}{61.71}\right)^2 \approx 1.1917\,x - \left(4.2278\times10^{-3}\right)x^2} \quad (2)$$

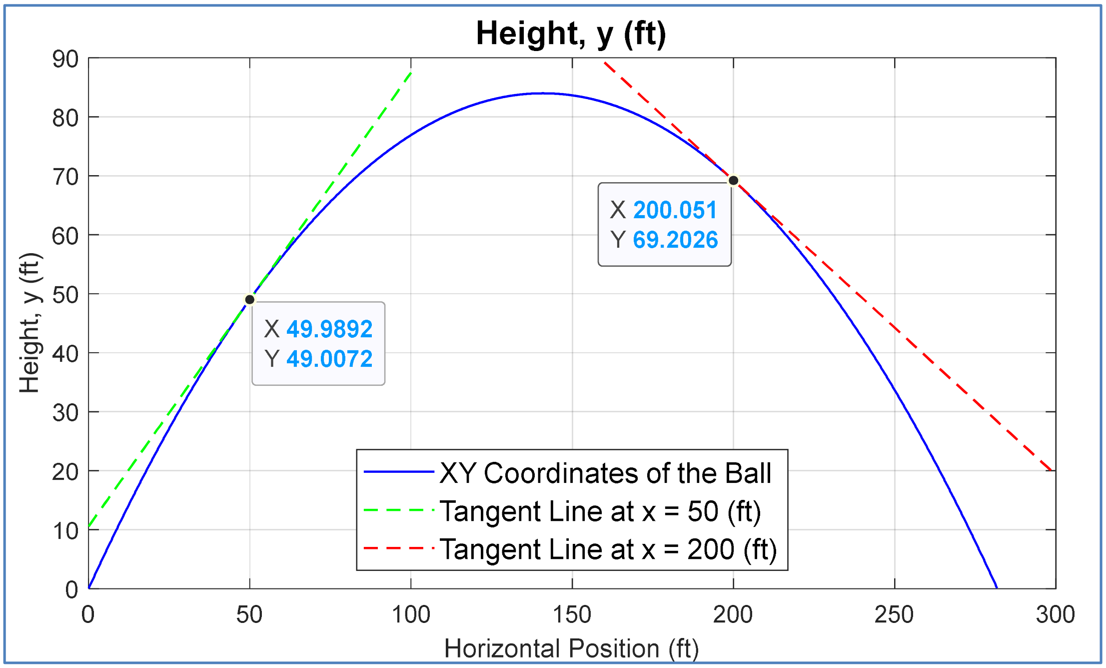


Figure 1. Height of Golf Ball as a Function of Distance, x

The ***velocity*** of the ball is in the direction of the tangent line.

<u>Find</u>: The ***derivative*** of the function $f(x)$ at a) $x_0 = 50$ (ft) and b) $x_0 = 200$ (ft). Use both graphical and analytical methods. Then find the ***equation*** of the ***tangent line***, and the angle between the ***velocity vector*** and the X-axis for each case.

Solution:

a) Derivative at $x_0 = 50$:
Graphical method:

Using the plot in Fig. 1: $$f'(x_0) = \frac{\Delta y}{\Delta x} \approx \frac{90-10}{100-0} = \frac{80}{100} = 0.8 \qquad (3)$$

Analytical method:

$$f(x_0) = 1.1917(50) - \left(4.2278\times10^{-3}\right)\left(50^2\right) = 59.585 - 10.5695 = 49.0155$$

$$\begin{aligned} f(x_0+h) &= 1.1917(50+h) - \left(4.2278\times10^{-3}\right)\left((50+h)^2\right) \\ &= 1.1917(50+h) - \left(4.2278\times10^{-3}\right)\left(50^2 + 100h + h^2\right) \\ &= \left[1.1917(50) - 4.2278\times10^{-3}\left(50^2\right)\right] + \left[1.1917(h) - 4.2278\times10^{-3}\left(100h + h^2\right)\right] \\ &= 49.0155 + \left(0.7689h - 4.2278\times10^{-3}\left(h^2\right)\right) \end{aligned}$$

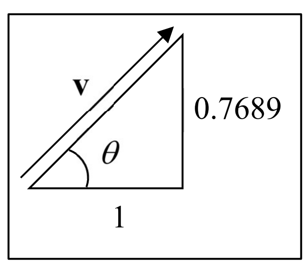
$$\begin{aligned} f'(x_0) &= \lim_{h\to0}\left(\frac{f(x_0+h) - f(x_0)}{h}\right) \\ &= \lim_{h\to0}\left(\frac{\left(49.0155 + 0.7689h - (4.2278\times10^{-3})h^2\right) - 49.0155}{h}\right) \\ &= \lim_{h\to0}\left(\frac{0.7689h - (4.2278\times10^{-3})h^2}{h}\right) \\ &= \lim_{h\to0}\left(0.7689 - (4.2278\times10^{-3})h\right) \\ &= 0.7689 \text{ (close to our graphical approximation)} \end{aligned} \qquad (4)$$

We can find the equation of the tangent line at $x_0 = 50$ using the ***point-slope form***:

$$\frac{y - y_0}{x - x_0} = m \quad \Rightarrow \quad \frac{y - 49.0155}{x - 50} = 0.7689 \quad \Rightarrow \quad y = 10.57 + 0.7689x \qquad (5)$$

The angle between the velocity vector and the X-axis is found using the inverse tangent function.

$$\theta(x_0) = \tan^{-1}(\Delta y/\Delta x) = \tan^{-1}(0.7689) = \begin{cases} 37.56 \text{ (deg)} \\ 0.6555 \text{ (rad)} \end{cases}$$



b) Derivative at $x_0 = 200$:

Graphical method:

Using the plot in Fig. 1: $\boxed{f'(x_0) = \dfrac{\Delta y}{\Delta x} \approx \dfrac{20-90}{300-160} = \dfrac{-70}{140} = -0.5}$ (6)

Analytical method:

$$\boxed{f(x_0) = 1.1917\,(200) - \left(4.2278\times10^{-3}\right)\left(200^2\right) = 238.34 - 169.112 = 69.228}$$

$$\begin{aligned} f(x_0+h) &= 1.1917\,(200+h) - \left(4.2278\times10^{-3}\right)\left((200+h)^2\right) \\ &= 1.1917(200+h) - \left(4.2278\times10^{-3}\right)\left(200^2 + 400h + h^2\right) \\ &= \left[1.1917(200) - 4.2278\times10^{-3}\left(200^2\right)\right] \\ &\quad + \left[1.1917(h) - 4.2278\times10^{-3}\left(400h + h^2\right)\right] \\ &= 69.228 - \left(0.4994\,h + 4.2278\times10^{-3}\left(h^2\right)\right) \end{aligned}$$

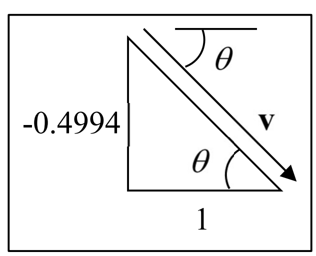
$$\begin{aligned} f'(x_0) &= \lim_{h\to0}\left(\frac{f(x_0+h) - f(x_0)}{h}\right) \\ &= \lim_{h\to0}\left(\frac{\left(69.228 - 0.4994\,h - (4.2278\times10^{-3})h^2\right) - 69.228}{h}\right) \\ &= \lim_{h\to0}\left(\frac{-0.4994\,h - (4.2278\times10^{-3})h^2}{h}\right) \\ &= \lim_{h\to0}\left(-0.4994 - (4.2278\times10^{-3})h\right) \\ &= -0.4994 \;\; \text{(again close to our graphical approximation)} \end{aligned} \qquad (7)$$

We can find the equation of the tangent line at $x_0 = 200$ by using the ***point-slope form***:

$$\boxed{\frac{y-y_0}{x-x_0} = m} \;\Rightarrow\; \boxed{\frac{y-69.228}{x-200} = -0.4994} \;\Rightarrow\; \boxed{y = 169.11 - 0.4994\,x} \qquad (8)$$

The angle between the velocity vector and the *X*-axis is found using the inverse tangent function.

$$\boxed{\theta(x_0) = \tan^{-1}(\Delta y/\Delta x) = \tan^{-1}(-0.4994) = \begin{cases} -26.54 \text{ (deg)} \\ -0.4632 \text{ (rad)} \end{cases}}$$



Example 2:

Given: The tangent line equation at $x_0 = 50$ for the golf ball trajectory is given by Eq. (5) to be

$$\boxed{y = 10.57 + 0.7689\,x}\,.$$

Find: Use this equation to generate approximate values for $f(x)$ in the range $30 \le x \le 70$. Compare these values with those found using the exact equation given in Eq. (2). For each value, calculate the percent error of the approximation.

Solution:

The approximate values, actual values, and percent error are summarized in the table below.

x	y_{approx}	y_{actual}	$\%\text{ error} = 100(y_{\text{approx}} - y_{\text{actual}})/y_{\text{actual}}$
30	33.6370	31.9460	5.29
35	37.4815	36.5304	2.60
40	41.3260	40.9035	1.03
45	45.1705	45.0652	0.23
50	49.0150	49.0155	≈ 0
55	52.8595	52.7544	0.20
60	56.7040	56.2819	0.75
65	60.5485	59.5980	1.59
70	64.3930	62.7028	2.70