

Introductory Control Systems

Armature Controlled DC Motor Transfer Functions

(Reference: Dorf and Bishop, Modern Control Systems, 9th Ed., Prentice-Hall, Inc. 2001)

- In an armature-current controlled DC motor, the field current i_f is held constant, and the armature current is controlled through the **armature voltage** V_a .
- The **motor torque** increases linearly with the armature current.

$$T_m = K_{ma} i_a$$

- K_{ma} is a **constant** that depends on a given motor. The **transfer function** from the input armature current to the resulting motor torque is

$$\frac{T_m(s)}{I_a(s)} = K_{ma} \quad (1)$$

- The **voltage/current relationship** for the armature side of the motor is

$$V_a = V_R + V_L + V_b = R_a i_a + L_a \left(\frac{di_a}{dt} \right) + V_b \quad (2)$$

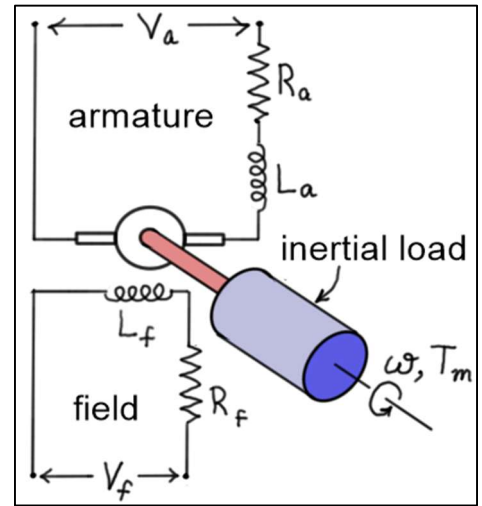
- Here, V_b represents the “**back-EMF**” induced by the rotation of the armature windings in a magnetic field. V_b is proportional to the motor speed ω

$$V_b(s) = K_b \omega(s)$$

K_b is the back-EMF coefficient.

- Taking Laplace transforms of Eq. (2) gives

$$V_a(s) - V_b(s) = (R_a + L_a s) I_a(s) \quad \text{or} \quad V_a(s) - K_b \omega(s) = (R_a + L_a s) I_a(s) \quad (3)$$



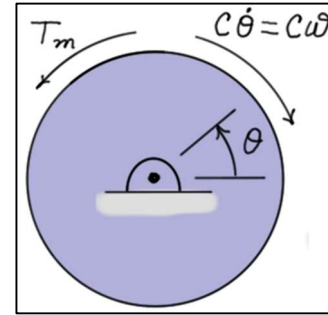
DC Motor

- An equation describing the **rotational motion** of the **inertial load** is found by **summing moments** about the motor shaft.

$$\sum M = T_m - c\omega = J\dot{\omega} \quad (\text{CCW positive})$$

or

$$\boxed{J\dot{\omega} + c\omega = T_m} \quad (4)$$



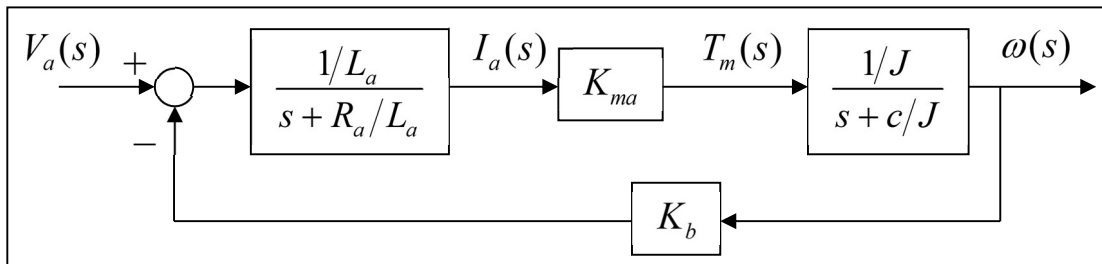
Free Body Diagram
of the Inertial Load

Eq. (4) is the differential equation of rotational motion, J is the **moment of inertia** of the load about the axis of rotation, and c is the **damping coefficient**.

- The **transfer function** from the input **motor torque** to **rotational speed** changes is found by applying **Laplace transforms** to Eq. (4),

$$\boxed{\frac{\omega(s)}{T_m(s)} = \frac{(1/J)}{s + (c/J)}} \quad (1^{\text{st}} \text{ order system}) \quad (5)$$

- Equations (1), (3) and (5) together can be represented by the **closed loop block diagram** shown below.



- **Block diagram reduction** gives the transfer function from the input **armature voltage** to the resulting **motor speed change**.

$$\boxed{\frac{\omega(s)}{V_a(s)} = \frac{(K_{ma}/L_a J)}{(s + R_a/L_a)(s + c/J) + (K_b K_{ma}/L_a J)}} \quad (2^{\text{nd}} \text{ order system}) \quad (6)$$

This is a second-order transfer function from armature voltage to motor speed changes.

- If the *time constant* of the *electrical circuit* is *much smaller* than the *time constant* of the *inertial load dynamics*, the transfer function of Eq. (6) can be reduced to a *first-order* transfer function. Namely,

$$\boxed{\frac{\omega}{V_a}(s) = \frac{K_{ma} / R_a J}{s + (cR_a + K_b K_{ma}) / R_a J}} \quad (1^{\text{st}} \text{ order system}) \quad (7)$$

- The *transfer function* from the input *armature voltage* to the resulting *angular position change* is found by multiplying Eqs. (6) and (7) by $\frac{1}{s}$.