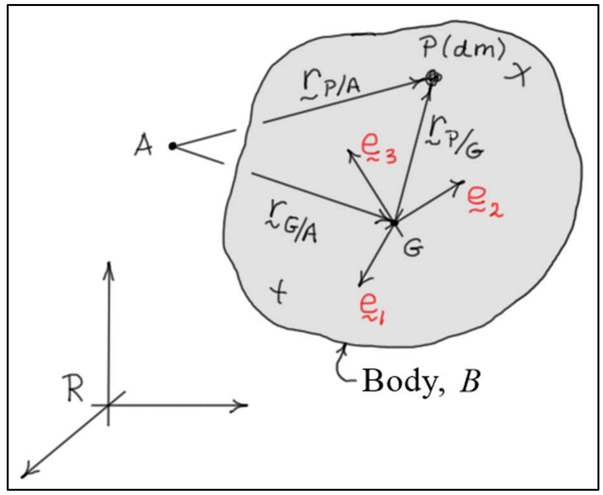
Intermediate Dynamics
Angular Momentum of a Rigid Body about an Arbitrary Point

The ***angular momentum*** of a rigid body B about its mass center G is defined as



$$\boxed{\underset{\sim}{H}_G = \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P\right) dm} \tag{1}$$

In previous notes, it was shown that $\underset{\sim}{H}_G$ can be written in terms of the body's moments and products of inertia and the body-fixed angular velocity components as follows.

$$\boxed{\underset{\sim}{H}_G = \left(I^G_{xx}\omega_1 - I^G_{xy}\omega_2 - I^G_{xz}\omega_3\right)\underset{\sim}{n}_1 + \left(-I^G_{xy}\omega_1 + I^G_{yy}\omega_2 - I^G_{yz}\omega_3\right)\underset{\sim}{n}_2 + \left(-I^G_{xz}\omega_1 - I^G_{yz}\omega_2 + I^G_{zz}\omega_3\right)\underset{\sim}{n}_3}$$

or

$$\boxed{\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B} \tag{2}$$

Here, $\underset{\approx}{I}_G$ represents the ***inertia dyadic*** (or matrix) of B about its mass-center G, and ${}^R\underset{\sim}{\omega}_B$ represents the ***angular velocity*** of the body.

The angular momentum of a rigid body about an ***arbitrary point*** A is similarly defined as

$$\boxed{\underset{\sim}{H}_A = \int_B \left(\underset{\sim}{r}_{P/A} \times {}^R\underset{\sim}{v}_P\right) dm} \tag{3}$$

Substituting $\underset{\sim}{r}_{P/A} = \underset{\sim}{r}_{G/A} + \underset{\sim}{r}_{P/G}$ into the integrand, distributing the cross product over the sum, and expanding gives

$$\begin{aligned}
\underset{\sim}{H}_A &= \int_B \left(\left(\underset{\sim}{r}_{G/A} + \underset{\sim}{r}_{P/G}\right) \times {}^R\underset{\sim}{v}_P\right) dm = \int_B \left(\underset{\sim}{r}_{G/A} \times {}^R\underset{\sim}{v}_P\right) dm + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P\right) dm \\
&= \underset{\sim}{r}_{G/A} \times \left(\int_B {}^R\underset{\sim}{v}_P \, dm\right) + \int_B \left(\underset{\sim}{r}_{P/G} \times {}^R\underset{\sim}{v}_P\right) dm \\
&= \underset{\sim}{r}_{G/A} \times \left(m\,{}^R\underset{\sim}{v}_G\right) \quad + \quad \underset{\sim}{H}_G
\end{aligned}$$

Or,

$$\boxed{\underset{\sim}{H}_A = \underset{\sim}{H}_G \; + \; \underset{\sim}{r}_{G/A} \times m\,{}^R\underset{\sim}{v}_G} \tag{4}$$

So, the ***angular momentum*** of the body about an ***arbitrary point*** A is equal to the ***angular momentum*** of the body about its ***mass-center*** G plus the ***moment*** of the body's ***linear momentum*** about A.

<u>Special Case: Motion about a Fixed-Point on the Body</u>

If some point O of the rigid body is ***fixed*** so that the body ***pivots*** about this point, the velocity of the mass center may be written as ${}^R\underset{\sim}{v}_G = {}^R\underset{\sim}{v}_O + {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O}$. Substituting this result into Eq. (4) above and combining terms, it can be shown that the angular momentum of the body about point O can be written as

$$\boxed{\underset{\sim}{H}_O = \underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\omega}_B} \qquad (5)$$

Here, $\underset{\approx}{I}_O$ represents the inertia dyadic (or matrix) about the fixed-point O.