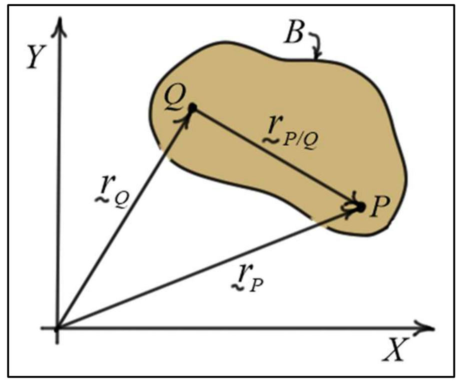
Elementary Dynamics
Relative Acceleration of Two Points Fixed on a Rigid Body

The figure depicts a rigid body B moving in two dimensions. The two points P and Q are ***fixed*** on B. At any instant of time, the position vector of P can be written as



$$\boxed{\underset{\sim}{r}_P = \underset{\sim}{r}_Q + \underset{\sim}{r}_{P/Q}}$$

Here, $\underset{\sim}{r}_{P/Q}$ is called the position vector of P relative to Q.

In previous notes, it was shown that this equation could be differentiated to find a relationship between the velocities of P and Q. It was shown

$$\boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \underset{\sim}{v}_{P/Q} = \underset{\sim}{v}_Q + \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/Q}\right)} \qquad \text{(\textbf{\textit{relative velocity}} equation)}$$

Here, $\underset{\sim}{v}_{P/Q}$ is the velocity of P relative to Q which is the ***velocity of P*** assuming ***Q is fixed*** and $\underset{\sim}{\omega}$ is the angular velocity of B. Similarly, the accelerations of points P and Q can be related by ***differentiating*** the relative velocity equation.

$$\underset{\sim}{a}_P = \frac{d}{dt}\left(\underset{\sim}{v}_Q + \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/Q}\right)\right) = \frac{d\underset{\sim}{v}_Q}{dt} + \frac{d}{dt}\left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/Q}\right) = \underset{\sim}{a}_Q + \left(\frac{d\underset{\sim}{\omega}}{dt} \times \underset{\sim}{r}_{P/Q}\right) + \left(\underset{\sim}{\omega} \times \frac{d\underset{\sim}{r}_{P/Q}}{dt}\right)$$

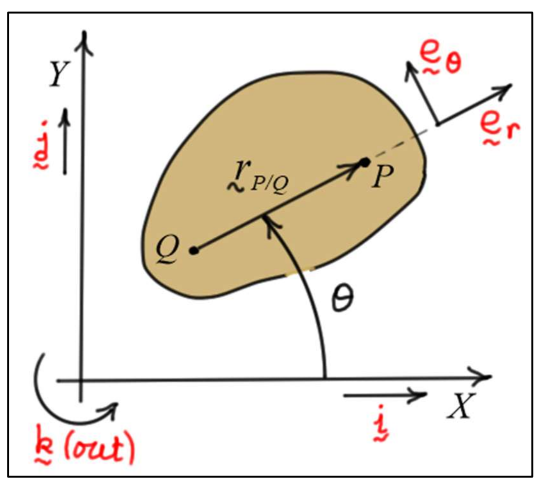
or

$$\boxed{\underset{\sim}{a}_P = \underset{\sim}{a}_Q + \underset{\sim}{a}_{P/Q} = \underset{\sim}{a}_Q + \left(\underset{\sim}{\alpha} \times \underset{\sim}{r}_{P/Q}\right) + \left(\underset{\sim}{\omega} \times \underset{\sim}{v}_{P/Q}\right) = \underset{\sim}{a}_Q + \left(\underset{\sim}{\alpha} \times \underset{\sim}{r}_{P/Q}\right) + \underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/Q}\right)}$$

Here, $\underset{\sim}{\alpha}$ is the angular acceleration of B and, in ***two dimensions***, the triple vector product $\underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/Q}\right) = -\omega^2 \underset{\sim}{r}_{P/Q}$. Thus, for two-dimensional motion,

$$\boxed{\underset{\sim}{a}_P = \underset{\sim}{a}_Q + \underset{\sim}{a}_{P/Q} = \underset{\sim}{a}_Q + \left(\underset{\sim}{\alpha} \times \underset{\sim}{r}_{P/Q}\right) - \omega^2 \underset{\sim}{r}_{P/Q}}$$

Here, $\underset{\sim}{a}_{P/Q}$ is the acceleration of P relative to Q, that is, the ***acceleration*** of P assuming Q is ***fixed***. It can be written in terms of unit vectors $\underset{\sim}{e}_r$ and $\underset{\sim}{e}_\theta$ as follows.



$$\boxed{\underset{\sim}{a}_{P/Q} = \left(\underset{\sim}{a}_{P/Q}\right)_t + \left(\underset{\sim}{a}_{P/Q}\right)_n = L\alpha\,\underset{\sim}{e}_\theta - L\omega^2 \underset{\sim}{e}_r}$$

Here, L is the ***distance*** between P and Q.