Introductory Control Systems
Proportional Position Control of a Spring-Mass-Damper (SMD)

- Fig. 1 shows a ***spring-mass-damper*** system with a ***force actuator*** for ***position control***. The spring has stiffness k, the damper has coefficient b, the block has mass m, and the position of the mass is measured by the variable x.

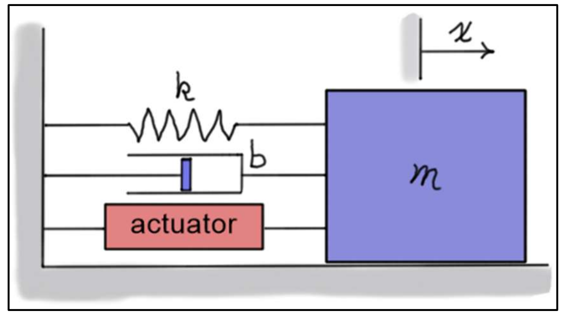
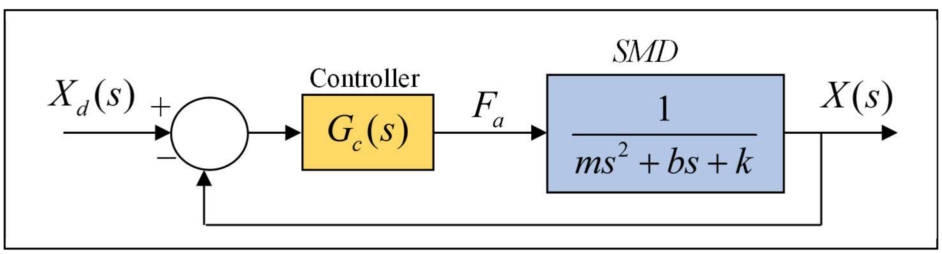


Figure 1. Spring-Mass-Damper System with Force Actuator

- The ***transfer function*** of the SMD: ***input*** is actuating force F_a and ***output*** is position x

$$\frac{X}{F_a}(s)=\frac{1}{ms^2+bs+k} \tag{1}$$

- This is a ***second-order*** transfer function that may be ***over-damped***, ***under-damped***, or ***critically damped*** depending on the values of the parameters m, b, and k.
- Assuming ***ideal actuator*** and ***sensor*** responses, the closed-loop ***position control*** of the SMD can be described using the following block diagram. Here, X_d represents the ***desired position***, X represents the ***actual position***, and $G_c(s)$ represents the transfer function of the ***controller***.



- If simple ***proportional control*** is used, then $G_c(s)=K$. Using ***block diagram reduction***, the transfer function for this case is found to be

$$\frac{X}{X_d}(s)=\frac{K}{ms^2+bs+(k+K)} \tag{2}$$

- The ***closed-loop*** system is also a ***second-order*** system, but it is ***not*** quite the same as the open-loop system. Using Eq. (2), the following observations can be made.

- **Observations**:
  1. The "stiffness" of the closed-loop system is $(k+K)>k$ for all positive gains K. However, the "mass" and "damping" coefficients are the ***same*** as for the open-loop SMD. This will give the closed-loop system a ***higher natural frequency*** ω_n.
  2. The ***damping ratio*** ζ of the closed-loop system is ***smaller*** than that of the open-loop system, because the product $2\zeta\omega_n = b/m$ is the ***same*** for both systems, and ω_n is ***higher*** for the closed-loop system.
  3. For a ***unit step command***, the final value of $x(t)$ is

$$\boxed{x_{ss} = \lim_{s\to 0}\left(\cancel{s}\cdot\frac{1}{\cancel{s}}\cdot\frac{K}{ms^2+bs+(K+k)}\right) = \frac{K}{K+k} < 1}$$

- The general conclusion here is that ***proportional control*** can be used to ***alter*** the system's response. The question is whether the altered response is an ***acceptable response***. To examine this question, consider the following example.

<u>Example</u>: $\boxed{m = 1 \text{ slug}}$, $\boxed{b = 8.8 \text{ (lb-s/ft)}}$, and $\boxed{k = 40 \text{ (lb/ft)}}$

- Using these values, the ***natural frequency*** and ***damping ratio*** of the open-loop system are

$\boxed{\omega_n = \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)}}$ and $\boxed{\zeta = \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7}$.

- The following table lists the ***natural frequencies***, ***damping ratios***, and ***final values*** of the closed-loop system to a ***unit step input*** for various values of controller parameter K.

Gain, K	Natural Frequency ω_n (rad/s), (Hz)	Damping Ratio ζ	Final Value to a Unit Step
100	11.83, 1.88	0.37	0.71
500	23.24, 3.7	0.19	0.93
1000	32.25, 5.13	0.14	0.96
2000	45.17, 7.19	0.1	0.98

- Note that the values shown in the table ***corroborate*** the observations listed above. Unfortunately, for this system, when the gain K is high enough to produce a final value close to one, the damping ratio is quite small. So, ***proportional control*** is not necessarily a good choice for this system. As will be seen later in these notes, the addition of ***integral*** and ***derivative*** terms to the controller make for a better closed-loop response.

- Fig. 2 below shows the ***closed loop step response*** of the closed-loop system for gains K of 100, 500, and 2000. Note, as expected, that as K ***increases***, the ***frequency of response*** and ***final value*** both ***increase***, and the ***damping ratio decreases***.

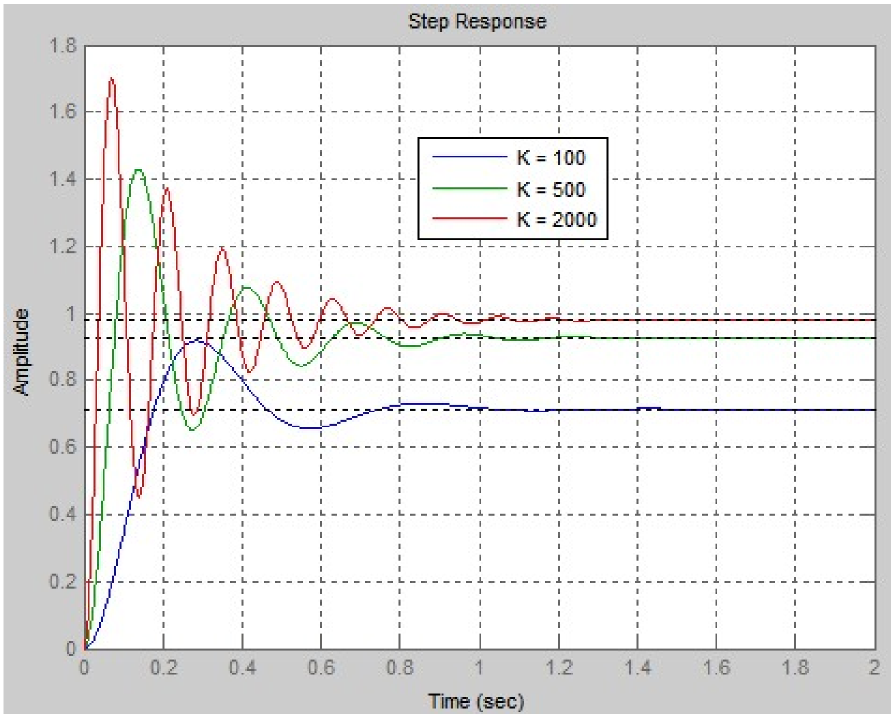


Figure 2. Step Response of the Closed-Loop System for Various Gains