

Elementary Engineering Mathematics

Applications of Derivatives in Statics, Mechanics of Materials

Example #1

Consider a **long slender beam** of length L with a **concentrated load** P acting at distance a from the left end. Due to this load, the beam experiences an **internal bending moment** $M(x)$ and **internal shearing force** $V(x)$. As presented in earlier notes, the bending moment is zero at both ends of the beam and rises linearly from there to a maximum value at $x = a$. The shearing force is the derivative of the bending moment.

$$V(x) = \frac{dM(x)}{dx} = M'(x)$$

Given: $P = 100$ (lbs), $L = 5$ (ft), $a = 3.5$ (ft),

$$\text{and } M_{\max} = abP/L$$

Find: (a) $M(x)$ for $0 \leq x \leq L$; (b) $V(x)$ for $0 \leq x \leq L$; and (c) plot the functions.

Solution: $M_{\max} = abP/L = (3.5)(1.5)100/5 = 105$ (ft-lb)

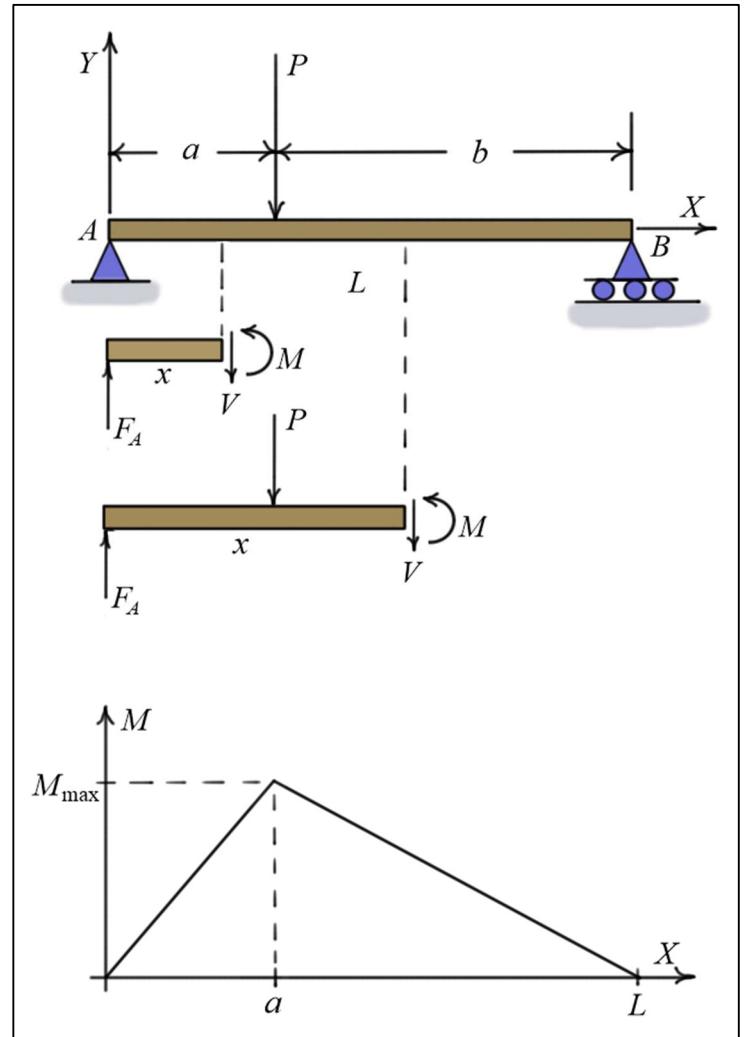
(a) For $(0 \leq x \leq a)$, the slope is $m = (105 - 0)/(3.5 - 0) = 30$ (ft-lb/ft).

$$M(x) = 30x \text{ (ft-lb)}$$

For $(a \leq x \leq L)$, the slope is $m = (0 - 105)/(5 - 3.5) = -70$ (ft-lb/ft). Using the point-slope form

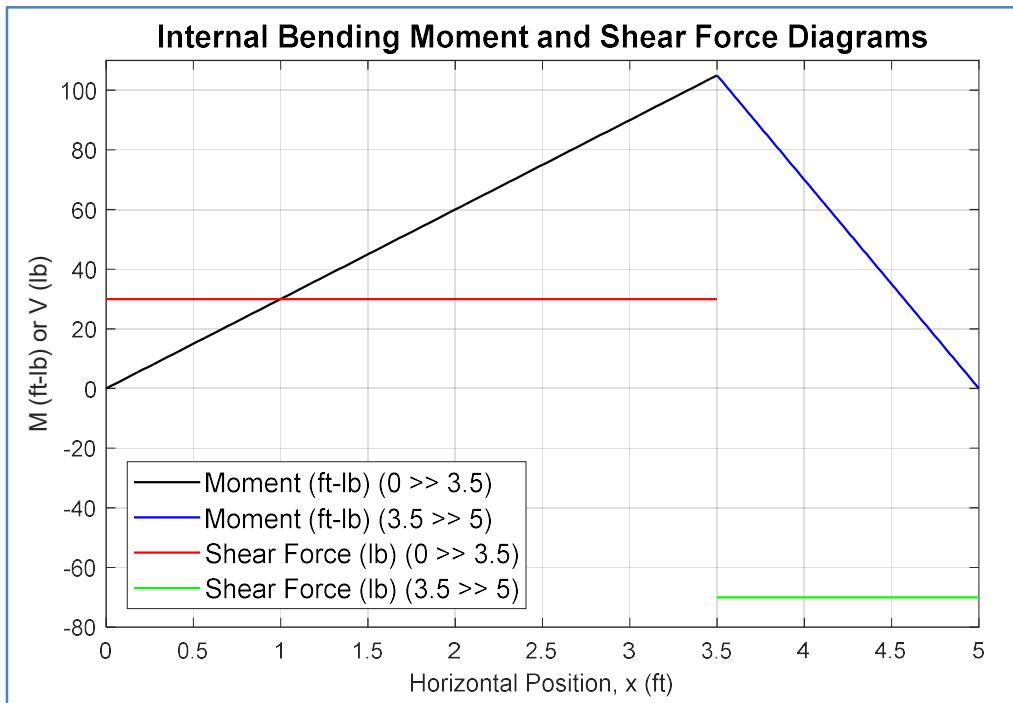
$$(M - 105) = -70(x - 3.5) \Rightarrow M(x) = 350 - 70x \text{ (ft-lb)}$$

(b) For $(0 \leq x \leq a)$, $V(x) = M'(x) = \frac{d}{dx}(30x) = 30$ (lb)



$$\text{For } (a \leq x \leq L), \boxed{V(x) = M'(x) = \frac{d}{dx}(350 - 70x) = \frac{d}{dx}(350) + \frac{d}{dx}(-70x) = -70 \text{ (lb)}}$$

(c) Plot of the shear force and bending moment along the beam.



Question: What is the value of $M'(x)$ at $x = 3.5$ (ft)?

Example 2:

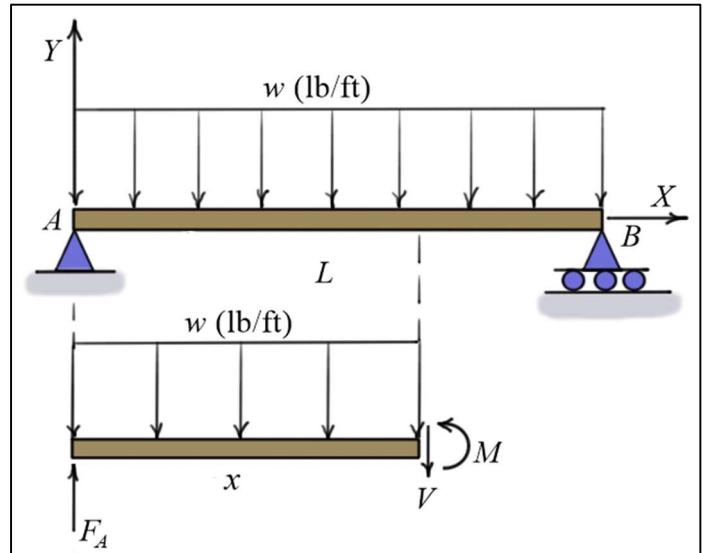
Given: $L = 10$ (ft), $w = 100$ (lb/ft), and

$$\boxed{M(x) = 500x - 50x^2 \text{ (ft-lb)}} \quad (0 \leq x \leq L)$$

Find: (a) shearing force $V(x)$; (b) maximum bending moment and its location; and (c) plot $M(x)$ and $V(x)$.

Solution:

(a) For $(0 \leq x \leq L)$



$$\boxed{V(x) = M'(x) = \frac{d}{dx}(500x - 50x^2) = \frac{d}{dx}(500x) + \frac{d}{dx}(-50x^2) = 500 - 100x \text{ (lb)}}$$

(b) Because the shearing force is continuous, the bending moment is a maximum (or minimum) either at an end of the beam or where the shear **zero**.

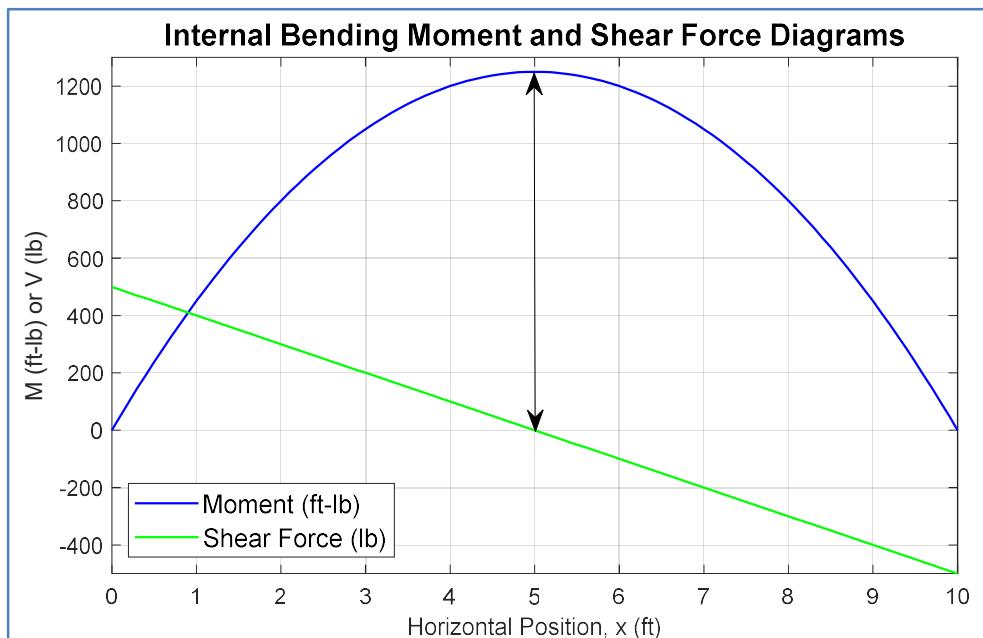
$$V(x) = M'(x) = 500 - 100x = 0 \Rightarrow x = 500/100 = 5 \text{ (ft)}$$

$$M(0) = M(L) = 0 \text{ and } M(x=5) = (500 \times 5) - (50 \times 5^2) = 1250 \text{ (ft-lb)} = M_{\max}$$

To verify that it is a maximum, check the sign of $M''(x)$:

$$M''(x) = \frac{d}{dx}(500 - 100x) = -100 < 0 \text{ (it is a **maximum**)}$$

(c) Plot of the shear force and bending moment along the beam.



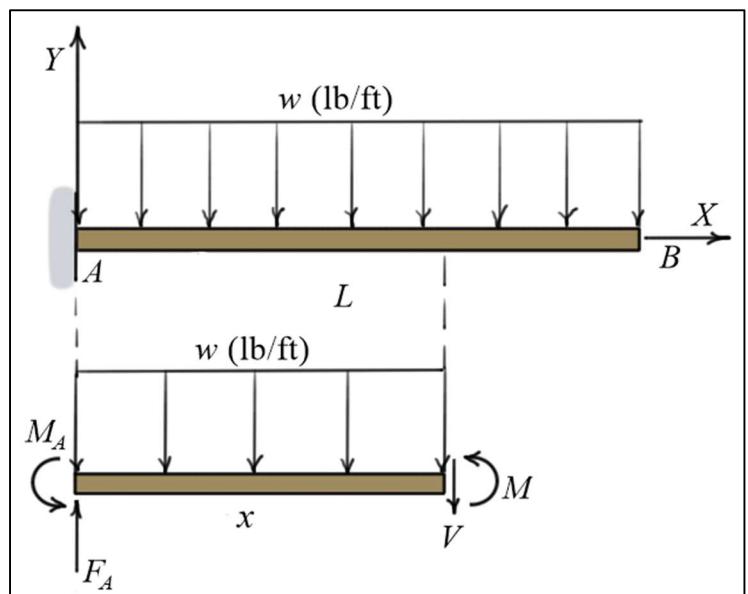
Example 3:

Consider a cantilevered beam with a **uniformly distributed load** of w (lb/ft). If the beam is cut at a distance x from the wall, we expose the internal **shearing force** V and **bending moment** M .

Given: $L = 10$ (ft), $w = 100$ (lb/ft), and

$$M(x) = -\frac{1}{2}wx^2 + wLx - \frac{1}{2}wL^2 \text{ (ft-lb)}$$

Find: (a) the shearing force $V(x)$; (b) the maximum bending moment and its location; and (c) plot $M(x)$ and $V(x)$.



Solution: Using the values for L and w , $M(x) = -50x^2 + 1000x - 5000$ (ft-lb)

(a)
$$V(x) = M'(x) = \frac{d}{dx}(-50x^2 + 1000x - 5000) = 1000 - 100x \text{ (lb)}$$

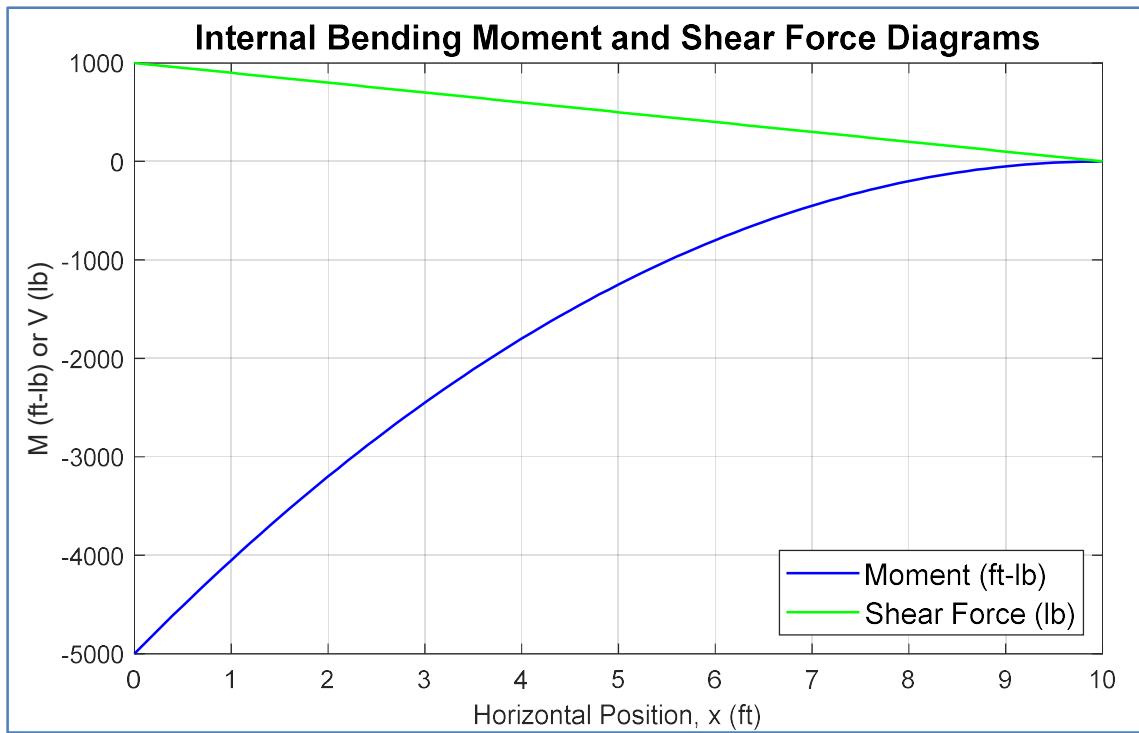
(b) Again, the **shearing force** is a **continuous** function, so the bending moment is a maximum (or minimum) **either** at an **end** of the beam or where the shear **zero**.

$$V(x) = M'(x) = 1000 - 100x = 0 \Rightarrow x = 1000/100 = 10 \text{ (ft)} \text{ (at the end)}$$

$$M(0) = -5000 \text{ (ft-lb)} \quad M(10) = 0 \text{ (ft-lb)} \Rightarrow M_{\max} = -5000 \text{ (ft-lb)}$$

In this case, the **maximum** occurs at the **left end** of the beam, and **not** where $M'(x) = 0$, because our concern is with the **absolute value** of the bending moment. We must design the beam to withstand 5000 (ft-lb) of bending moment, not zero.

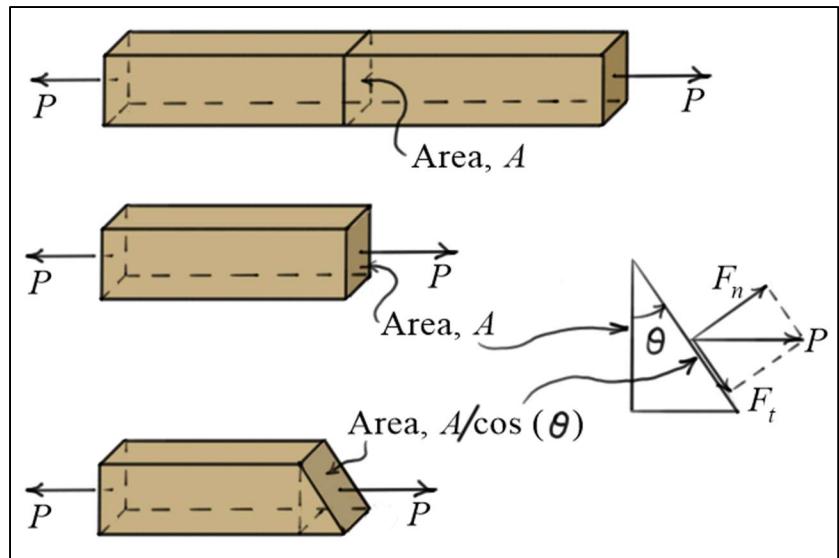
(c) Plot of the shear force and bending moment along the beam.



Example 4:

Consider a bar with rectangular cross-sectional area A and applied force P as shown. Because A is perpendicular (or normal) to the direction of P , the material on A experiences **normal stress** only and is defined as

$$\boxed{\sigma = P/A}$$



Now consider a plane at an angle θ to the vertical. Since this plane is not normal to P , the material along this plane experiences both **normal stress** and **shear stress**.

The normal stress σ is defined as the ratio of the normal force and the area. The shear stress τ is defined as the ratio of the tangential force and the area.

$$\boxed{\sigma = \frac{F_n}{A/\cos(\theta)} = \frac{P \cos(\theta)}{A/\cos(\theta)} = \left(\frac{P}{A}\right) \cos^2(\theta)}$$

$$\boxed{\tau = \frac{F_t}{A/\cos(\theta)} = \frac{P \sin(\theta)}{A/\cos(\theta)} = \left(\frac{P}{A}\right) \sin(\theta) \cos(\theta)}$$

In a simple tension test, such as that described above, **brittle materials** tend to fail due to excessive **normal stress**, and **ductile materials** tend to fail due to excessive **shear stress**.

By thinking of the normal and shear stresses as functions of the cut angle θ , we can find which planes experience the **highest** normal and shear stresses. We can find maxima and minima by setting $d\sigma/d\theta = 0$ and $d\tau/d\theta = 0$ and then solving for the angle θ . Using the product and chain rules gives

$$\boxed{d\sigma/d\theta = \frac{d}{d\theta} \left[(P/A) \cos^2(\theta) \right] = (P/A) (2 \cos(\theta)) (-\sin(\theta)) = -(2P/A) \sin(\theta) \cos(\theta)}$$

$$\boxed{d^2\sigma/d\theta^2 = \frac{d}{d\theta} \left[-(2P/A) \sin(\theta) \cos(\theta) \right] = (2P/A) (\sin^2(\theta) - \cos^2(\theta))}$$

$$\boxed{d\tau/d\theta = \frac{d}{d\theta} \left[(P/A) \sin(\theta) \cos(\theta) \right] = (P/A) [\cos^2(\theta) - \sin^2(\theta)] = (P/A) \cos(2\theta)}$$

$$\begin{aligned}
 \frac{d^2\tau}{d\theta^2} &= \frac{d}{d\theta} \left[(P/A) \cos(2\theta) \right] = (P/A) \left[(-\sin(2\theta))(2) \right] \\
 &= (-2P/A) \sin(2\theta)
 \end{aligned}$$

Setting the derivatives to zero and considering $0 \leq \theta < \pi/2$, we get the following results.

Stress	Angle, θ	1 st Derivative	2 nd Derivative	Type
σ	0	0	negative	maximum
τ	$\pi/4$ (rad) = 45°	0	negative	maximum

So, **brittle materials** will be more likely to **fail** on a plane **normal to** the load, and **ductile materials** will be more likely to fail on a **45° plane**.