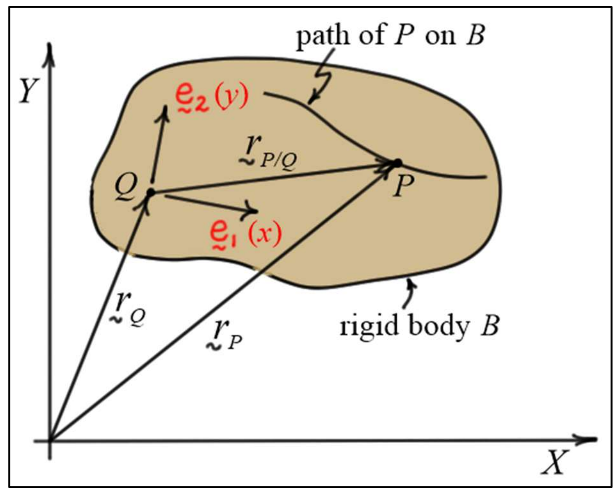
Elementary Dynamics
Point Moving on a Rigid Body (Sliding Contact)

The relative motion of ***two points fixed on a rigid body*** can be calculated using the relative velocity and relative acceleration equations. These equations can be used to analyze many systems such as slider-crank and four-bar mechanisms. However, many systems have ***sliding contacts*** on ***rotating bodies***. These systems cannot be analyzed using the relative velocity or relative acceleration equations. We need a new set of kinematical equations for these systems.

Velocity of a Point Moving on a Rotating Body



Consider the rigid body B shown in the diagram. Point P ***moves*** on B, while point Q is ***fixed*** on B. The unit vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ along the x and y directions are fixed in and rotate with B. The position vector of P can be written as

$$\underset{\sim}{r}_P = \underset{\sim}{r}_Q + \underset{\sim}{r}_{P/Q} = \underset{\sim}{r}_Q + \left(b_1\underset{\sim}{e}_1 + b_2\underset{\sim}{e}_2\right).$$

The velocities of P and Q can be related by differentiating this expression

$$\underset{\sim}{v}_P = \frac{d\underset{\sim}{r}_P}{dt} = \frac{d\underset{\sim}{r}_Q}{dt} + \frac{d}{dt}\left(b_1\underset{\sim}{e}_1 + b_2\underset{\sim}{e}_2\right) = \underset{\sim}{v}_Q + \left(\dot{b}_1\underset{\sim}{e}_1 + \dot{b}_2\underset{\sim}{e}_2\right) + \left(b_1\dot{\underset{\sim}{e}}_1 + b_2\dot{\underset{\sim}{e}}_2\right)$$

where

$$\dot{b}_1\underset{\sim}{e}_1 + \dot{b}_2\underset{\sim}{e}_2 \triangleq \underset{\sim}{v}_{P\text{rel}} \qquad \text{(the velocity of } P \textbf{ \textit{relative to}} \text{ body } B\text{)}$$

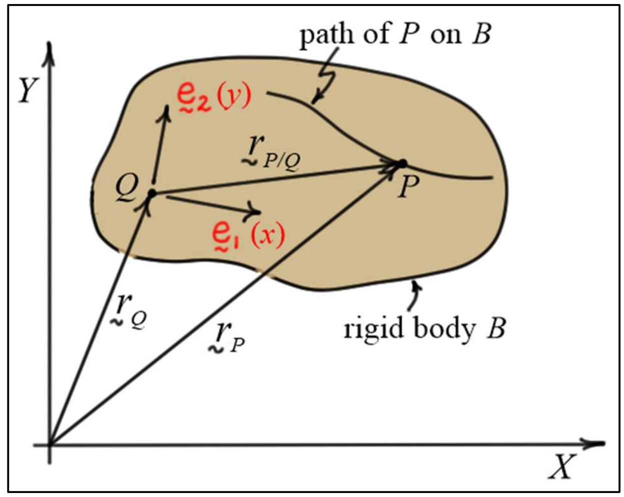
$$b_1\dot{\underset{\sim}{e}}_1 + b_2\dot{\underset{\sim}{e}}_2 = b_1\left(\underset{\sim}{\omega}_B \times \underset{\sim}{e}_1\right) + b_2\left(\underset{\sim}{\omega}_B \times \underset{\sim}{e}_2\right) = \underset{\sim}{\omega}_B \times \left(b_1\underset{\sim}{e}_1 + b_2\underset{\sim}{e}_2\right) = \underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}$$

Here, $\underset{\sim}{\omega}_B$ is the angular velocity of B. Substituting into the expression for $\underset{\sim}{v}_P$ gives

$$\boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \underset{\sim}{v}_{P\text{rel}} + \underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}}$$

Note this equation is like the relative velocity equation, but it also includes the velocity of P relative to body B. Note also that some texts use a slightly different notation for the velocity of P relative to B. They write $\underset{\sim}{v}_{P\text{rel}} = \left(\underset{\sim}{v}_{P/Q}\right)_{xy}$ or $\underset{\sim}{v}_{P\text{rel}} = {}^B\underset{\sim}{v}_P$.

Acceleration of a Point Moving on a Rotating Body



The accelerations of points P and Q can be related by differentiating again as follows

$$\underset{\sim}{a}_P = \frac{d\underset{\sim}{v}_P}{dt} = \frac{d\underset{\sim}{v}_Q}{dt} + \frac{d\underset{\sim}{v}_{\text{Prel}}}{dt} + \frac{d}{dt}\left(\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right)$$

where

$$\begin{aligned}\frac{d\underset{\sim}{v}_{\text{Prel}}}{dt} &= \frac{d}{dt}\left(\dot{b}_1\underset{\sim}{e}_1 + \dot{b}_2\underset{\sim}{e}_2\right)\\ &= \left(\ddot{b}_1\underset{\sim}{e}_1 + \ddot{b}_2\underset{\sim}{e}_2\right) + \left(\underset{\sim}{\omega}_B \times \underset{\sim}{v}_{\text{Prel}}\right)\\ &= \underset{\sim}{a}_{\text{Prel}} + \left(\underset{\sim}{\omega}_B \times \underset{\sim}{v}_{\text{Prel}}\right)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right) &= \left(\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{P/Q}\right) + \left(\underset{\sim}{\omega}_B \times \frac{d\underset{\sim}{r}_{P/Q}}{dt}\right)\\ &= \left(\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{P/Q}\right) + \underset{\sim}{\omega}_B \times \left(\underset{\sim}{v}_{\text{Prel}} + \left(\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right)\right)\end{aligned}$$

Substituting these results into the equation for $\underset{\sim}{a}_P$ gives

$$\boxed{\underset{\sim}{a}_P = \underset{\sim}{a}_Q + \underset{\sim}{a}_{\text{Prel}} + 2\left(\underset{\sim}{\omega}_B \times \underset{\sim}{v}_{\text{Prel}}\right) + \left(\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{P/Q}\right) + \underset{\sim}{\omega}_B \times \left(\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right)}$$

In two dimensions, this expression can be reduced to

$$\boxed{\underset{\sim}{a}_P = \underset{\sim}{a}_Q + \underset{\sim}{a}_{\text{Prel}} + 2\left(\underset{\sim}{\omega}_B \times \underset{\sim}{v}_{\text{Prel}}\right) + \left(\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{P/Q}\right) - \omega_B^2\,\underset{\sim}{r}_{P/Q}}$$

Note this equation has two more terms than the relative acceleration equation. It has $\underset{\sim}{a}_{\text{Prel}}$ the acceleration of P ***relative*** to the ***body***, and it also has $2\left(\underset{\sim}{\omega}_B \times \underset{\sim}{v}_{\text{Prel}}\right)$ which is called the ***Coriolis acceleration***. Note as before that some texts use different notation for the acceleration of P relative to B. They write $\underset{\sim}{a}_{\text{Prel}} = \left(\underset{\sim}{a}_{P/Q}\right)_{xy}$ or $\underset{\sim}{a}_{\text{Prel}} = {}^{B}\underset{\sim}{a}_P$.