

Introductory Control Systems

Characteristics of Open-Loop and Closed-Loop Systems

Important Control System Characteristics

- **Sensitivity** of system response to *parametric variations* can be **reduced**
- **Transient** and **steady-state responses** of a system can be **altered**
- **Steady-state error** can be **reduced**
- **Response** of system to **disturbances** (disturbance response) can be **lowered**

The effects of a control system on the overall response of a dynamic system can be **positive** or **negative**. It is the **responsibility** of the **analyst** to **design** the control system so it has **beneficial effects** on system performance.

Parametric Sensitivity: Concept

From a **mathematical perspective**, a dynamic system will have identical responses to repeated applications of the same input. The step response of a system as calculated by MATLAB, for example, is always the same (unless the transfer function is altered).

For **real systems**, however, this is **not** the case. **Each time** a real system is subjected to an input, its response **will vary**. The variations may be **small random fluctuations** producing the same average response, or they may be **large fluctuations** producing very different responses. For example, the effects of **friction** and **damping** can easily vary during the day-to-day operation of a system.

To **compare** the **effects** of **variations** on the response of **open-loop** and **closed-loop** systems, consider the systems shown in Fig. 1. In each system, the transfer function $G(s)$ has been **changed** to $G(s) + \Delta G(s)$. Here, $\Delta G(s)$ represents **small changes** to $G(s)$.

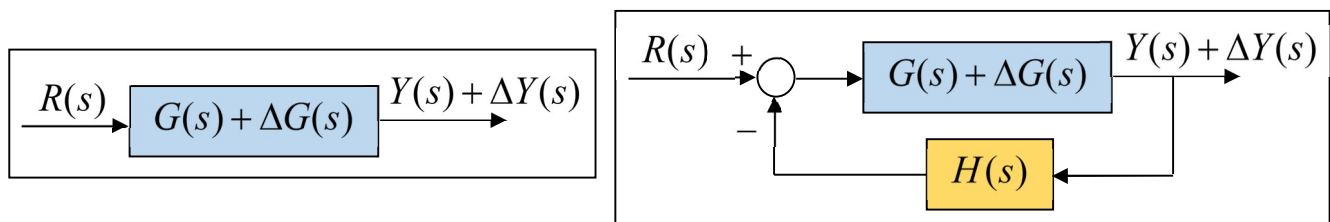


Figure 1. Open-Loop and Closed-Loop Systems with Plant Variations

The **algebraic equation** associated with the **open-loop system** is

$$Y(s) + \Delta Y(s) = (G(s) + \Delta G(s))R(s) = G(s)R(s) + \Delta G(s)R(s)$$

So, **changes** in the output can be written as

$$\boxed{\Delta Y(s) = \Delta G(s)R(s)} \quad (1)$$

The plant changes are clearly **passed directly to the system output**.

The **algebraic equation** associated with the **closed-loop system** is

$$\begin{aligned} Y(s) + \Delta Y(s) &= \left(\frac{G(s) + \Delta G(s)}{1 + (G(s) + \Delta G(s))H(s)} \right) R(s) \\ &= \left(\frac{G(s)}{1 + (G(s) + \Delta G(s))H(s)} \right) R(s) + \left(\frac{\Delta G(s)}{1 + (G(s) + \Delta G(s))H(s)} \right) R(s) \\ &\approx \left(\frac{G(s)}{1 + GH(s)} \right) R(s) + \left(\frac{\Delta G(s)}{1 + GH(s)} \right) R(s) \\ &\approx Y(s) + \left(\frac{\Delta G(s)}{1 + GH(s)} \right) R(s) \end{aligned}$$

So, **changes** in the output can be approximated as

$$\boxed{\Delta Y(s) \approx \left(\frac{\Delta G(s)}{1 + GH(s)} \right) R(s)} \quad (2)$$

From this result it clear that the amount of $\Delta G(s)$ that is passed to the output depends on the magnitude of the **loop** (or open-loop) **transfer function** $GH(s)$. The **larger** the **magnitude** of $GH(s)$, the **less** changes in $G(s)$ will affect the system response.

Sensitivity: Calculation

The **calculation** of **sensitivity** is done more formally using derivatives. Specifically, S_{α}^T the sensitivity of a system with transfer function $T(s)$ to changes in a parameter α is defined as follows.

$$\boxed{S_{\alpha}^T = \frac{\alpha}{T} \left(\frac{\partial T}{\partial \alpha} \right)} \quad (3)$$

Generally, **sensitivity** is a function of s , and hence a **function of frequency**.

If the **magnitude** of S_α^T is **between zero and one** ($0 < |S_\alpha^T| < 1$), then the effects of $\Delta G(s)$ will be **lowered** (i.e. 10% changes in $G(s)$ will result in less than 10% changes in the response). However, if the **magnitude** of S_α^T is **greater than one** ($|S_\alpha^T| > 1$), then the effects of $\Delta G(s)$ will be **magnified** (i.e. 10% changes in $G(s)$ will result in greater than 10% changes in the response).

To gain some **general insight** into the issue of sensitivity for a simple closed-loop system (Fig. 2), consider the sensitivity of the system transfer function $T(s) = \frac{G}{1+GH}$ to **bulk changes** in the transfer functions $G(s)$ or $H(s)$.

$$\boxed{S_G^T = \frac{G}{T} \left(\frac{\partial T}{\partial G} \right)} \quad \text{and} \quad \boxed{S_H^T = \frac{H}{T} \left(\frac{\partial T}{\partial H} \right)}$$

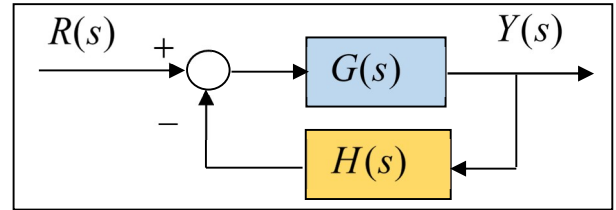


Figure 2. Simple Closed Loop System

Using these **definitions** and the **quotient rule** for **differentiation** gives

$$\boxed{S_G^T = \frac{G}{T} \frac{\partial}{\partial G} \left(\frac{G}{1+GH} \right) = \cancel{G} \left(\frac{1+\cancel{GH}}{\cancel{G}} \right) \left(\frac{1(1+\cancel{GH}) - \cancel{GH}}{(1+\cancel{GH})(1+GH)} \right) = \frac{1}{1+GH}} \quad (4)$$

$$\boxed{S_H^T = \frac{H}{T} \left(\frac{\partial T}{\partial H} \right) = H \left(\frac{1+\cancel{GH}}{\cancel{G}} \right) \left(\frac{-\cancel{G}}{(1+\cancel{GH})(1+GH)} \right) = \frac{-GH}{1+GH}} \quad (5)$$

Eq. (4) indicates that as $|GH(s)|$ is **increased**, the effects on the response of the system to changes in $G(s)$ are **lowered**. This is the same conclusion that was drawn from Eq. (2). However, Eq. (5) indicates that as $|GH(s)|$ is **increased**, the effects on the response of the system to changes in $H(s)$ are **passed directly to the output**, that is, $S_H^T \approx 1$.

Note also that for an **open-loop system** with transfer function $T(s) = G(s)$,

$$\boxed{S_G^T = S_G^G = \frac{G}{G} \left(\frac{\partial G}{\partial G} \right) = 1}$$

This again indicates that changes in the plant will be passed **directly** to the output.

Although Eqs. (4) and (5) provide **general insights** into the **usefulness** of **closed-loop control**, Eq. (3) is used to determine the sensitivity to specific system parameters. For this reason, Eq. (3) provides more **detailed information** about the system at hand. For example, in previous notes, the **closed-loop transfer function** for **proportional position control** of a **spring-mass-damper system** was found to be

$$T(s) = \frac{K}{ms^2 + bs + (k + K)}$$

The **sensitivity** of this system to changes in the damping and spring stiffness parameters can be calculated as follows.

$$\begin{aligned} S_b^T &= \frac{b}{T} \frac{\partial T}{\partial b} = b \left(\frac{ms^2 + bs + (k + K)}{K} \right) \frac{\partial}{\partial b} \left(\frac{K}{ms^2 + bs + (k + K)} \right) \\ &= b \left(\frac{\cancel{ms^2 + bs + (k + K)}}{K} \right) \left(\frac{-\cancel{K}s}{(\cancel{ms^2 + bs + (k + K)}) (ms^2 + bs + (k + K))} \right) \\ &\Rightarrow S_b^T = \frac{-bs}{ms^2 + bs + (k + K)} \end{aligned}$$

$$\begin{aligned} S_k^T &= \frac{k}{T} \frac{\partial T}{\partial k} = k \left(\frac{ms^2 + bs + (k + K)}{K} \right) \frac{\partial}{\partial k} \left(\frac{K}{ms^2 + bs + (k + K)} \right) \\ &= k \left(\frac{\cancel{ms^2 + bs + (k + K)}}{K} \right) \left(\frac{-\cancel{K}}{(\cancel{ms^2 + bs + (k + K)}) (ms^2 + bs + (k + K))} \right) \\ &\Rightarrow S_k^T = \frac{-k}{ms^2 + bs + (k + K)} \end{aligned}$$

Control of Transient Response

In previous notes the **block diagram** for the **open-loop response** of an **armature-controlled DC motor** was given. If the **electrical response** of the motor is much **faster** than the **mechanical speed changes**, then the **time dependence** of the **circuitry** can be **ignored**. Under these conditions, the block diagram reduces to that shown in Fig 3.

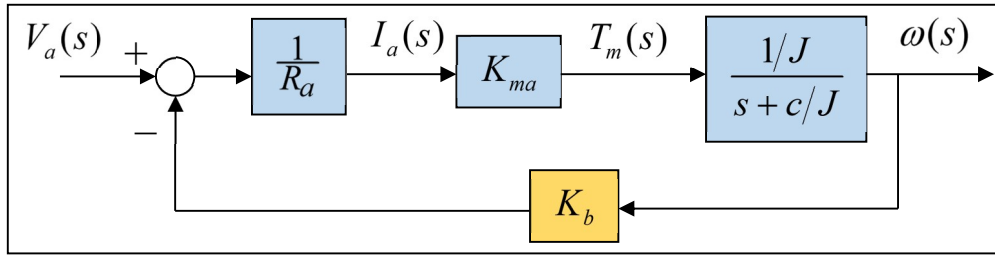


Figure 3. Block Diagram of an Armature-Controlled DC Motor

As before, the **input** to the motor is the **armature voltage** $V_a(s)$ and the **output** is the **angular velocity** (speed) of the motor. Using block diagram reduction, the **open-loop transfer function** for this system is found to be

$$\frac{\omega(s)}{V_a(s)} = \frac{K^*}{s + a^*} \quad \left\{ \begin{array}{l} K^* = K_{ma}/R_a J \\ a^* = (R_a c + K_b K_{ma})/R_a J \end{array} \right. \quad (6)$$

The parameters K^* and a^* are **constants** that depend on the **motor characteristics**, **inertial load**, and **damping coefficient**. Note that the value of a^* **determines how quickly the motor responds** when a step increase in voltage is applied to the motor.

To study the **effects of feedback** on transient response, consider **proportional, closed-loop control** of the DC motor as shown in Fig 4. The **input** to the system is the **desired angular velocity** $\omega_d(s)$ and the **output** of the system is the **actual angular velocity** $\omega(s)$. The parameter K_t is the **calibration constant of the tachometer** that relates changes in angular velocity to changes in voltage. The signal $E(s)$ represents a tachometer voltage error, and the parameter K_a is the proportional gain.

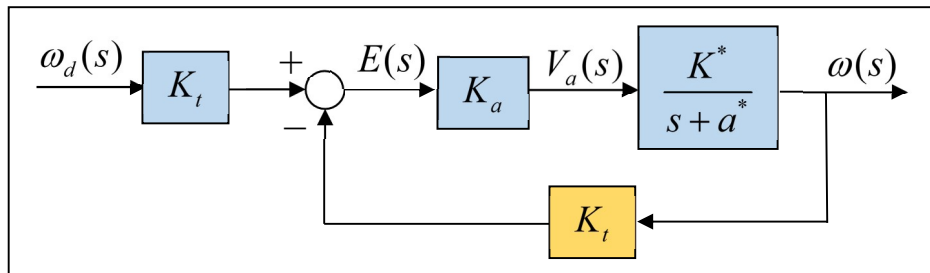


Figure 4. Proportional Control of a DC Motor Using a Tachometer

The **transfer function** of the **closed-loop system** is found using block diagram reduction as follows.

$$\frac{\omega}{\omega_d}(s) = \frac{K_t \hat{K}}{s + \hat{a}} \quad \begin{cases} \hat{K} = K_a K^* \\ \hat{a} = a^* + K_a K^* K_t \end{cases} \quad (7)$$

The parameter \hat{a} *determines the speed of response* of the *closed-loop, speed control system*. The value of \hat{a} can be *increased* by *increasing* the *proportional gain* K_a . Note, however, that if the value of K_a is increased *too much*, the voltage input to the motor may become *too large*, potentially damaging the motor. Physical limitations such as these are often not part of the mathematical model, so the analyst must be aware of them.

Control of Steady-State Error

Consider again the closed-loop speed control system of Fig. 4. To track the *error* in the system as it responds to a commanded speed change, the error signal $E(s)$ is taken to be the output of the system as shown in Fig. 5.

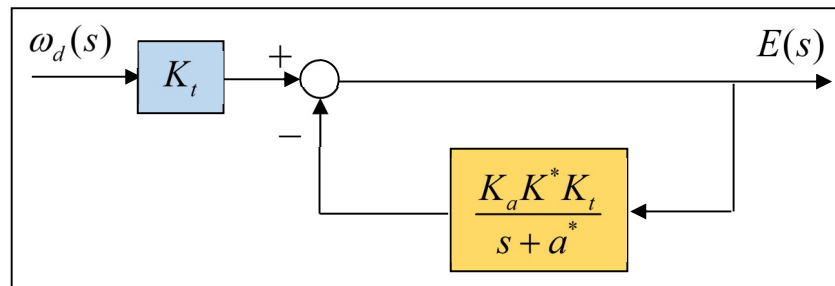


Figure 5. Error of a DC Motor Speed Control System

Using block diagram reduction, the system *error transfer function* is found to be

$$\frac{E}{\omega_d}(s) = \frac{K_t(s + a^*)}{s + a^* + K_a K^* K_t} = \frac{K_t(s + a^*)}{s + \hat{a}} \quad (8)$$

Using the *final value theorem*, the *steady-state error* to a unit step, speed change command is found to be

$$e_{ss} = \lim_{s \rightarrow 0} \left(s \cdot \frac{E}{\omega_d} \cdot \frac{1}{s} \right) = \frac{K_t a^*}{\hat{a}} \quad (\text{steady-state error}) \quad (9)$$

This result shows that the proportional control gain *affects* the steady-state error K_a . As the value of K_a is *increased*, the value of \hat{a} is *increased* and e_{ss} the *steady-state error* is *decreased*.

Control of Disturbance Response

Consider the block diagram of an armature-controlled DC motor with a *disturbance torque* $T_D(s)$ as shown in Fig. 6. It is assumed that the disturbance torque *reduces* the torque generated by the motor under ideal conditions.

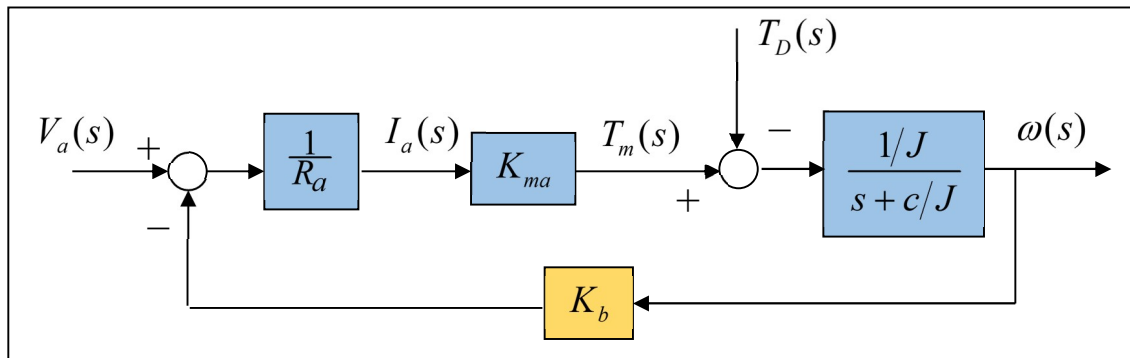


Figure 6. Block Diagram of an Armature-Controlled DC Motor with Disturbance

To study the effect of the disturbance on the response of this system, the *disturbance transfer function* must be found. One way to do this is to move the disturbance to the left-most summing block as shown in Fig. 7.

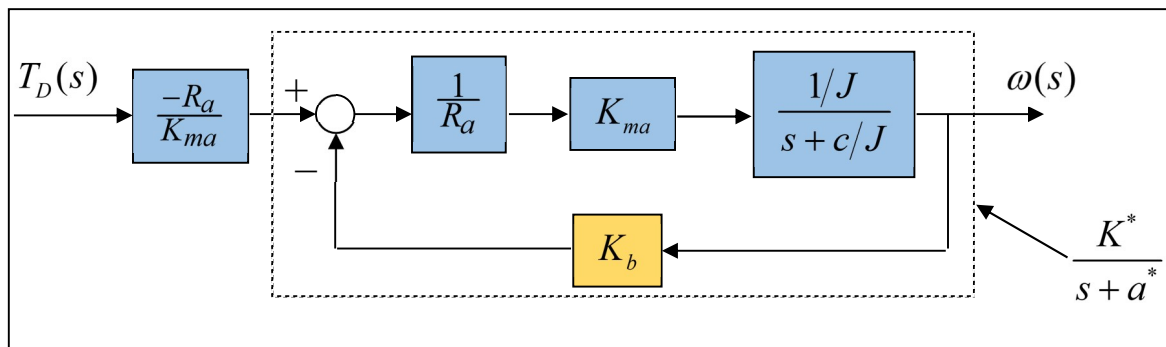


Figure 7. Block Diagram of a DC Motor with Disturbance Input Only

The *disturbance transfer function* for the *open-loop system* is then identified to be

$$\frac{\omega}{T_D}(s) = \frac{-R_a K^* / K_{ma}}{s + a^*} \quad (\text{open loop system}) \quad (10)$$

Following this same approach, the **disturbance transfer function** of the **closed-loop, speed control system** of Fig. 4 can be found. In that case, to move the disturbance to the left-most summing block, the disturbance must be additionally moved over the proportional gain block as shown in Fig. 8.

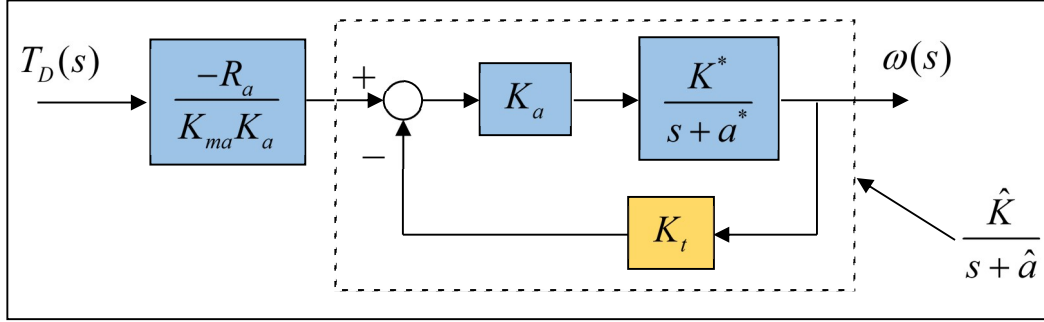


Figure 8. Disturbance Input in Speed Control System of a DC Motor

Using block diagram reduction, the **disturbance transfer function** for the **closed-loop, speed control system** is found to be

$$\frac{\omega}{T_D}(s) = \frac{-R_a \hat{K} / K_{ma} K_a}{s + \hat{a}} = \frac{-1/J}{s + \hat{a}} \quad (\text{closed-loop, speed control system}) \quad (11)$$

The **steady-state angular velocity change** of the motor to a **unit step disturbance torque** is found using the final value theorem.

$$\left(\omega_{ss} \right)_{T_D} = \lim_{s \rightarrow 0} \left(s \cdot \frac{\omega}{T_D} \cdot \frac{1}{s} \right) = \frac{-1}{J \hat{a}} \quad (\text{closed-loop, speed control system}) \quad (12)$$

These last two results indicate that as the proportional gain K_a is **increased**, the disturbance response **decays faster**, and the **steady-state angular velocity change** is **decreased**. Both are positive effects.